

SOME RESULTS OF HAMILTONIAN HOMEOMORPHISMS ON CLOSED ASPHERICAL SURFACES

JIAN WANG

ABSTRACT. On closed symplectically aspherical manifolds, Schwarz proved a classical result that the action function of a nontrivial Hamiltonian diffeomorphism is not constant by using Floer homology. In this article, we generalize Schwarz's theorem to the C^0 -case on closed aspherical surfaces. Our methods involve the theory of transverse foliations for dynamical systems of surfaces inspired by Le Calvez and its recent progresses. As an application, we prove that the contractible fixed points set (and consequently the fixed points set) of a nontrivial Hamiltonian homeomorphism is not connected. Furthermore, we obtain that the growth of the action width of a Hamiltonian homeomorphism increases at least linearly, and that the group of Hamiltonian homeomorphisms of \mathbb{T}^2 and the group of area preserving homeomorphisms isotopic to the identity of Σ_g ($g > 1$) are torsion free, where Σ_g is a closed orientated surface with genus g . Finally, we will show how the C^1 -Zimmer's conjecture on surfaces deduces from C^0 -Schwarz's theorem.

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1. INTRODUCTION

1.1. Background. The famous Gromov-Eliashberg Theorem, that the group of symplectic diffeomorphisms is C^0 -closed in the full group of diffeomorphisms, makes us interested in defining a symplectic homeomorphism as a homeomorphism which is a C^0 -limit of symplectic diffeomorphisms. This becomes a central theme of what is now called “ C^0 -symplectic topology”. There is a family of problems in symplectic topology that are interesting to be extended to the continuous analogs of classical smooth objects of the symplectic world (see, e.g., [4, 5, 19, 20, 21, 24, 25, 31, 34, 38]). In the theme of C^0 -symplectic topology, there are many questions still open, e.g., the C^0 -flux conjecture (see [1, 27, 30]) and the simplicity of the group of Hamiltonian homeomorphisms of surfaces (see [7, 31]).

Another noteworthy rigidity phenomenon is the Zimmer program which attracted many mathematicians to work (see, e.g., [2, 8, 11, 14, 15, 24, 32]). A central conjecture of Zimmer program [42] predicts that lattices in simple Lie groups of rank n do not act volume-preserving faithfully on compact manifolds of dimension less than n .

Suppose that (M, ω) is a symplectic manifold. Let $I = (F_t)_{t \in \mathbb{R}}$ be a Hamiltonian flow on M with $F_0 = \text{Id}_M$ and $F_1 = F$. When M is compact, among the properties of F , one may notice that it preserves the volume form $\omega^n = \omega \wedge \cdots \wedge \omega$ and that the “rotation vector” $\rho_{M,I}(\mu)$ (see Section 2.3) of the finite measure μ induced by ω^n vanishes. Let M be a closed oriented surface with genus $g \geq 1$. In this case, M is a closed aspherical surface with the property $\pi_2(M) = 0$. Let $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on M , that is, I is a continuous path in $\text{Homeo}(M)$ with $F_0 = \text{Id}_M$. We suppose that its time-one map F preserves the measure μ induced by ω . It is well known that the condition

$\rho_{M,I}(\mu) = 0$ is equivalent to the fact that the homeomorphism F is in the C^0 -closure of $\text{Ham}(M, \omega)$. In this sense, we call such I a *Hamiltonian isotopy* and such F a *Hamiltonian homeomorphism*. In this article, we carry out some foundational studies of Hamiltonian homeomorphisms (and a more general notion) on closed aspherical surfaces. We also show the link between Zimmer's conjecture on surfaces and our work which is of independent interest.

Let (M, ω) be a symplectic manifold with $\pi_2(M) = 0$. Suppose that $H : \mathbb{R} \times M \rightarrow \mathbb{R}$, one-periodic in time, is the Hamiltonian function generating the flow I . Denote by $\text{Fix}_{\text{Cont}, I}(F)$ the set of contractible fixed points of F , that is, $x \in \text{Fix}_{\text{Cont}, I}(F)$ if and only if x is a fixed point of F and the oriented loop $I(x) : t \mapsto F_t(x)$ defined on $[0, 1]$ is contractible on M . The classical action function is defined, up to an additive constant, on $\text{Fix}_{\text{Cont}, I}(F)$ as follows

$$(1.1) \quad \mathcal{A}_H(x) = \int_{D_x} \omega - \int_0^1 H(t, F_t(x)) dt,$$

where $x \in \text{Fix}_{\text{Cont}, I}(F)$ and $D_x \subset M$ is any 2-simplex with $\partial D_x = I(x)$. The following deep result [37] was proved by using Floer homology with the real filtration induced by the action function.

Theorem 1.1 (Schwarz). *Let (M, ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Let $I = (F_t)_{t \in \mathbb{R}}$ be a Hamiltonian flow on M with $F_0 = \text{Id}_M$ and $F_1 = F$ generated by a Hamiltonian function H . Assume that $F \neq \text{Id}_M$. Then there are $x, y \in \text{Fix}_{\text{Cont}, I}(F)$ such that $\mathcal{A}_H(x) \neq \mathcal{A}_H(y)$.*

Let M be a closed oriented surface with genus $g \geq 1$ and F be the time-one map of an identity isotopy I on M . We denote by $\text{Homeo}(M)$ (resp. $\text{Diff}(M)$, $\text{Diff}^1(M)$) the set of homeomorphisms (resp. diffeomorphisms, C^1 -diffeomorphisms) of M . Denote by $\mathcal{M}(F)$ the set of Borel finite measures on M that are invariant by F and have no atoms on $\text{Fix}_{\text{Cont}, I}(F)$. Through the WB-property [39] (see Definition 1.3 below), the classical action of Hamiltonian diffeomorphism has been generalized to the case of Hamiltonian homeomorphism (and to more general cases) [39, page 86] (or see [40]):

Theorem 1.2. *Let $F \in \text{Homeo}(M)$ be the time-one map of an identity isotopy I on M . Suppose that $\mu \in \mathcal{M}(F)$ and $\rho_{M, I}(\mu) = 0$. In each of the following cases:*

- $F \in \text{Diff}(M)$ (not necessarily C^1);
- I satisfies the WB-property and the measure μ has full support;
- I satisfies the WB-property and the measure μ is ergodic,

an action function L_μ can be defined, which generalizes the classical one given in Eq. 1.1.

The contributions of this paper can be summarized as following:

- In the classical case, one can prove that the action function is a constant on a connected set of contractible fixed points by Sard's theorem. In each of the generalized cases given in Theorem 1.2, we prove that this property still holds (see Proposition 1.4). Our method is purely topological.
- Given the generalized action function, one may ask whether Schwarz's theorem is still true. We show in this article that it is true in the second case of Theorem 1.2 but no longer true when the measure μ has no full support even if $F \in \text{Diff}(M)$ (see Theorem 1.5).
- As applications of Proposition 1.4 and Theorem 1.5, we obtain that the contractible fixed points set (and consequently the fixed points set) of a nontrivial Hamiltonian homeomorphism is not connected. We emphasize that this is merely a 2-dimension phenomenon. Indeed, Buhovsky et al. [5] have recently constructed a Hamiltonian homeomorphism with a single fixed point on any closed symplectic manifold of

dimension at least four. However, in the classical C^1 -case this property always holds when (M^{2n}, ω) ($n \geq 1$) is a closed symplectically aspherical manifold by Schwarz's theorem. It seems to us that one can not obtain the result in the C^0 -case on dimension two through C^0 -approximation by Hamiltonian diffeomorphisms.

- We generalize Polterovich's result [32] on the growth of the action width to the C^0 -case, based on which we obtain that the groups $\text{Hameo}(\mathbb{T}^2, \mu)$ and $\text{Homeo}_*(\Sigma_g, \mu)$ ($g > 1$) (see below for the notations) are torsion free, where μ is a measure with full support.
- We give an alternative proof of the C^1 -Zimmer's conjecture on surfaces when the measure is a Borel finite measure with full support from C^0 -Schwarz's theorem.

This paper extends our unpublished manuscripts [39, 40] with substantial additional contents. A preceding work of this paper [40] is under review of a journal. Please refer to [39] or [40] for further details and related results.

1.2. Statement of results.

Before stating our main results, let us recall the WB-property and B-property.

Let M be a surface homeomorphic to the complex plane \mathbb{C} and let $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on M . For every two different fixed points z and z' of F_1 , the *linking number* $i_I(z, z') \in \mathbb{Z}$ is the degree of the map $\xi : \mathbf{S}^1 \rightarrow \mathbf{S}^1$ defined by

$$\xi(e^{2i\pi t}) = \frac{h \circ F_t(z') - h \circ F_t(z)}{|h \circ F_t(z') - h \circ F_t(z)|},$$

where $h : M \rightarrow \mathbb{C}$ is a homeomorphism. The linking number is independent of h .

Let F be the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on a closed oriented surface M of genus $g \geq 1$, and \tilde{F} be the time-one map of the lifted identity isotopy $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]}$ of I on the universal cover \tilde{M} of M . Assume that $\pi : \tilde{M} \rightarrow M$ is the covering map. Denote by Δ (resp. $\tilde{\Delta}$) the diagonal of $\text{Fix}_{\text{Cont}, I}(F) \times \text{Fix}_{\text{Cont}, I}(F)$ (resp. $\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})$).

We write by $\text{Homeo}_*(M)$ the identity component of the topological space of $\text{Homeo}(M)$ for the compact-open topology. When $g > 1$, it is well known that the fundamental group $\pi_1(\text{Homeo}_*(M))$ is trivial [17]. It implies that any two identity isotopies $I, I' \subset \text{Homeo}_*(M)$ with fixed endpoints are homotopic. Hence, I is unique up to homotopy, which implies that \tilde{F} is uniquely defined and independent of the choice of the isotopy from Id_M to F . When $g = 1$, \tilde{F} depends on the isotopy I since $\pi_1(\text{Homeo}_*(M)) \simeq \mathbb{Z}^2$ [16].

Note that the universal cover \tilde{M} is homeomorphic to \mathbb{C} . We define the *linking number* $i(\tilde{F}; \tilde{z}, \tilde{z}')$ for each pair $(\tilde{z}, \tilde{z}') \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ as

$$(1.2) \quad i(\tilde{F}; \tilde{z}, \tilde{z}') = i_{\tilde{I}}(\tilde{z}, \tilde{z}').$$

Definition 1.3. We say that I satisfies the *weak boundedness property* at $\tilde{a} \in \text{Fix}(\tilde{F})$, written WB-property at \tilde{a} , if there exists a positive number $N_{\tilde{a}}$ such that $|i(\tilde{F}; \tilde{a}, \tilde{b})| \leq N_{\tilde{a}}$ for all $\tilde{b} \in \text{Fix}(\tilde{F}) \setminus \{\tilde{a}\}$. We say that I satisfies the *weak boundedness property*, denoted WB-property, if it satisfies the weak boundedness property at every $\tilde{a} \in \text{Fix}(\tilde{F})$. Let $\tilde{X} \subseteq \text{Fix}(\tilde{F})$. We say that I satisfies the *boundedness property on \tilde{X}* , written B-property on \tilde{X} , if there exists a positive number $N_{\tilde{X}}$ such that $|i(\tilde{F}; \tilde{a}, \tilde{b})| \leq N_{\tilde{X}}$ for all $(\tilde{a}, \tilde{b}) \in \tilde{X} \times \text{Fix}(\tilde{F})$ with $\tilde{a} \neq \tilde{b}$. We say that I satisfies the *boundedness property*, denoted B-property, if $\tilde{X} = \text{Fix}(\tilde{F})$.

Obviously, the B-property implies the WB-property. It has been proved that the WB-property is satisfied if $F \in \text{Diff}(M)$ and that the B-property is satisfied if $F \in \text{Diff}^1(M)$

[39]. Moreover, the set of all WB-property points of I , that is, the set

$$\{\tilde{a} \in \text{Fix}(\tilde{F}) \mid I \text{ satisfies the WB-property at } \tilde{a}\}$$

is shown dense in $\text{Fix}(\tilde{F})$ [26]. In Lemma 2.7 below, we prove that I satisfies the B-property if the number of the connected components of $\text{Fix}_{\text{Cont},I}(F)$ is finite.

We say that a homeomorphism F is μ -symplectic if $\mu \in \mathcal{M}(F)$ has full support. An identity isotopy I is μ -Hamiltonian if the time-one map F is μ -symplectic and $\rho_{M,I}(\mu) = 0$. A homeomorphism F is μ -Hamiltonian if there exists a μ -Hamiltonian isotopy I such that the time-one map of I is F . The main results of this article are summarized as follows.

Proposition 1.4. *Under the hypotheses of Theorem 1.2, the action function defined in Theorem 1.2 is a constant on each connected component of $\text{Fix}_{\text{Cont},I}(F)$.*

Theorem 1.5. *Let F be the time-one map of a μ -Hamiltonian isotopy I . If I satisfies the WB-property and $F \neq \text{Id}_M$, the action function defined in Theorem 1.2 is not constant.*

Theorem 1.5 is a generalization of Schwarz's theorem on closed oriented surfaces. The main tools we use in its proof are the theory of transverse foliations for dynamical systems of surfaces inspired by Le Calvez [24, 25] and its recent progress [22].

Recall the classical version of Arnold conjecture for surface homeomorphisms due to Matsumoto [28] (see also [24]): any Hamiltonian homeomorphism has at least three contractible fixed points (see Theorem 5.1 below).

As a consequence of Proposition 1.4 and Theorem 1.5, we have the following theorem:

Theorem 1.6. *Let F be the time-one map of a μ -Hamiltonian isotopy I . If the set $\text{Fix}_{\text{Cont},I}(F)$ is connected, then F must be Id_M . In particular, if $\text{Fix}(F)$ is connected, then F must be Id_M .*

Proof. By Theorem 5.1, $\text{Fix}_{\text{Cont},I}(F) \neq \emptyset$. Moreover, if the set $\text{Fix}_{\text{Cont},I}(F)$ is connected, the isotopy must satisfy the WB-property according to Lemma 2.7. Therefore, the action function is well defined by Theorem 1.2. The conclusion follows from Proposition 1.4 and Theorem 1.5. Note that the connectedness of $\text{Fix}(F)$ implies that $\text{Fix}_{\text{Cont},I}(F) = \text{Fix}(F)$ because $\text{Fix}_{\text{Cont},I}(F)$ is an open and closed subset of $\text{Fix}(F)$. \square

If $F \neq \text{Id}_M$, Theorem 1.6 implies that the number of connected components of the set $\text{Fix}_{\text{Cont},I}(F)$ is at least 2, which is optimal by the following example.

Example 1.7. Let μ be the measure induced by the area form ω and D be a topological closed disk on M . Up to a diffeomorphism, we may suppose that D is the closed unit Euclidean disk and that $\omega|_D = dx \wedge dy$. Let us consider the polar coordinate for D with the center $z_0 = (0, 0)$. Consider the following isotopy $(F_t)_{t \in [0,1]}$ on M defined as follows

$$\begin{aligned} F_t : D &\rightarrow D \\ (r, \theta) &\mapsto (r, \theta + 2\pi r t), \end{aligned}$$

and $F_t|_{M \setminus D} = \text{Id}_{M \setminus D}$ for all $t \in [0, 1]$. Obviously, $\rho_{M,I}(\mu) = 0$ and $\text{Fix}_{\text{Cont},I}(F)$ has exactly two connected components: $\{z_0\}$ and $M \setminus \text{Int}(D)$, where $\text{Int}(D)$ is the interior of D .

By Theorem 1.6 and Theorem 5.1, if $\text{Fix}_{\text{Cont},I}(F)$ has exactly two connected components, its cardinality must be infinite.

Remark that a measure with full support is essential for Theorem 1.5 and Theorem 1.6. Without such condition, Theorem 1.5 can not hold as illustrated by Example 8.3 and 8.4 (Section 8). In the case where $M = \mathbb{T}^2$, Example 8.3 is also a counter-example of Theorem 1.6 if the measure is without full support. When the genus of M is more than two, one can choose an identity isotopy on M with exactly one contractible fixed point z (such isotopy exists by Lefschetz-Nielsen's formula) and the Dirac measure δ_z .

Denote by $\mathcal{IS}_*(M)$ the group of all identity isotopies $I = (F_t)_{t \in [0,1]}$ on M , where the composition is given by Equation 2.1 (Section 2.1)¹. We say that two identity isotopies $(F_t)_{t \in [0,1]}, (G_t)_{t \in [0,1]} \in \mathcal{IS}_*(M)$ are homotopic with fixed extremities if $F_1 = G_1$ and there exists a continuous map $[0, 1]^2 \rightarrow \text{Homeo}(M)$, $(t, s) \mapsto H_{t,s}$ such that $H_{0,s} = \text{Id}_M$, $H_{1,s} = F_1 = G_1$, $H_{t,0} = F_t$ and $H_{t,1} = G_t$.

Under the same hypotheses as Theorem 1.2, we define the *action spectrum of I* (up to an additive constant):

$$\sigma(I) = \{L_\mu(z) \mid z \in \text{Fix}_{\text{Cont}, I}(F)\} \subset \mathbb{R},$$

and the following *action width of I* :

$$\text{width}(I) = \sup_{x, y \in \sigma(I)} |x - y|.$$

It turns out that the action spectrum $\sigma(I)$ (and hence $\text{width}(I)$) is invariant by conjugation in $\text{Homeo}^+(M, \mu)$, where $\text{Homeo}^+(M, \mu)$ is the subgroup of $\text{Homeo}(M)$ whose elements preserve the measure μ and the orientation (see [39, Corollary 4.6.14]). Moreover, the action function L_μ only depends on the homotopic class with fixed endpoints of I (see Proposition 3.8 below), so do $\sigma(I)$ and $\text{width}(I)$. Observing that $\pi_1(\text{Homeo}_*(\Sigma_g)) \simeq \{0\}$ ($g > 1$), and that $\pi_1(\text{Homeo}_*(\mathbb{T}^2)) \simeq \mathbb{Z}^2$ and $\rho_{\mathbb{T}^2, I}(\mu) = 0$, we can simply write $\sigma(F)$ (resp. $\text{width}(F)$) instead of $\sigma(I)$ (resp. $\text{width}(I)$).

For any $q \geq 1$, we define an identity isotopy I^q on M : $I^q(z) = \prod_{k=0}^{q-1} I(F^k(z))$ for $z \in M$ (see Equation 2.1 in Section 2.1 for details). Under the hypotheses of Theorem 1.2, for every two distinct contractible fixed points a and b of F , the following iteration formula [39, Corollary 4.7.3] holds: $I_\mu(I^q; a, b) = qI_\mu(I; a, b)$ for all $q \geq 1$, where $I_\mu(I; a, b) = L_\mu(I; b) - L_\mu(I; a)$.

We follow the conventions of Polterovich [32]: given two positive sequences $\{a_n\}$ and $\{b_n\}$, we write $a_n \succeq b_n$ if there is $c > 0$ such that $a_n \geq cb_n$ for all $n \in \mathbb{N}$, and $a_n \sim b_n$ if $a_n \succeq b_n$ and $a_n \preceq b_n$. Based on Theorem 1.5 and the iteration formula above, we have the following conclusion which is a generalization of Proposition 2.6. A in [32].

Proposition 1.8. *Let F be the time-one map of a μ -Hamiltonian isotopy I . If I satisfies the WB-property and $F \neq \text{Id}_M$, then $\text{width}(F^n) \succeq n$.*

In fact, we define the action functions not only for the “Hamiltonian case: $\rho_{M, I}(\mu) = 0$ ” (the double quotation marks means that the isotopy I is not truly Hamiltonian if the measure μ has no full measure by the definition), but also for the “non-Hamiltonian case: $\rho_{M, I}(\mu) \neq 0$ ” (see Corollary 3.6 below for details). Recall that \tilde{F} is the time-one map of the lifted identity isotopy of I to \tilde{M} . In the non-Hamiltonian case, we define the action function l_μ on the set $\text{Fix}(\tilde{F})$ (see Corollary 3.6). When M is hyperbolic, we have the following theorem which is a generalization of Theorem 2.1. C in [32].

Theorem 1.9. *Let $F \in \text{Homeo}_*(M) \setminus \{\text{Id}_M\}$ be the time-one map of an identity isotopy I on a closed oriented surface M with $g > 1$. If F is μ -symplectic and I satisfies the WB-property, then the action function l_μ is not constant.*

Similar to $\sigma(I)$ and $\text{width}(I)$, we can define the following *lifted action spectrum of I* (up to an additive constant):

$$\tilde{\sigma}(I) = \{l_\mu(z) \mid z \in \text{Fix}(\tilde{F})\} \subset \mathbb{R},$$

¹ Usually, there is another definition of the composition as follows: for any two identity isotopies $I = (F_t)_{t \in [0,1]}, I' = (F'_t)_{t \in [0,1]} \in \mathcal{IS}_*(M)$, we define $I \circ I' = (F_t \circ F'_t)_{t \in [0,1]}$. However, the two new identity isotopies by such two definitions are homotopic with fixed extremities.

and *lifted action width* of I :

$$\widetilde{\text{width}}(I) = \sup_{x, y \in \widetilde{\sigma}(I)} |x - y|.$$

As before, $\widetilde{\sigma}(I)$ and $\widetilde{\text{width}}(I)$ are invariant by conjugation in $\text{Homeo}^+(M, \mu)$ and merely depend on the homotopic class with fixed endpoints of I . When $M = \Sigma_g$ ($g > 1$), we can simply write $\widetilde{\sigma}(F)$ (resp. $\widetilde{\text{width}}(F)$) instead of $\widetilde{\sigma}(I)$ (resp. $\widetilde{\text{width}}(I)$) since $\pi_1(\text{Homeo}_*(\Sigma_g))$ is trivial.

For each two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , a similar iteration formula holds (see [39, Corollary 4.6.12]): $i_\mu(I^q; \tilde{a}, \tilde{b}) = qi_\mu(I; \tilde{a}, \tilde{b})$ for all $q \geq 1$, where $i_\mu(I; \tilde{a}, \tilde{b}) = l_\mu(I; \tilde{b}) - l_\mu(I; \tilde{a})$. By Theorem 1.9 and this iteration formula, we also have

Proposition 1.10. *Let $F \in \text{Homeo}_*(M) \setminus \{\text{Id}_M\}$ be the time-one map of an identity isotopy I on a closed oriented surface M with $g > 1$. If F is μ -symplectic and I satisfies the WB-property, then $\widetilde{\text{width}}(F^n) \succeq n$.*

We fix a Borel finite measure μ which has a full support and has no atoms on M (e.g., the measure μ induced by the area form ω). Obviously, the sets $\text{Homeo}(M)$ and $\text{Homeo}_*(M)$ form groups (the operation is the composition of the maps). Denote by $\text{Homeo}_*(M, \mu)$ the subgroup of $\text{Homeo}_*(M)$ whose elements preserve the measure μ . Denote by $\text{Hameo}(M, \mu)$ the subset of $\text{Homeo}_*(M, \mu)$ whose elements are μ -Hamiltonian. It has been proved that $\text{Hameo}(M, \mu)$ forms a group [10].

Denote by $\mathcal{IS}_*(M, \mu)$ the subgroup of $\mathcal{IS}_*(M)$ whose element $(F_t)_{t \in [0,1]} \in \mathcal{IS}_*(M)$ satisfies $(F_1)_*\mu = \mu$. The homotopic relation is an equivalence relation on $\mathcal{IS}_*(M)$ (resp. $\mathcal{IS}_*(M, \mu)$). Denote the set of equivalence classes by $\mathcal{H}_*(M)$ (resp. $\mathcal{H}_*(M, \mu)$). It turns out that $\mathcal{H}_*(M)$ and $\mathcal{H}_*(M, \mu)$ are groups. Indeed, $\mathcal{H}_*(M)$ (resp. $\mathcal{H}_*(M, \mu)$) is the universal covering space of $\text{Homeo}_*(M)$ (resp. $\text{Homeo}_*(M, \mu)$) [7, Section 5].

Given $I \in \mathcal{H}_*(M, \mu)$, if the isotopy I does not satisfy the WB-property (Definition 1.3), then there must exist three fixed points \tilde{a}, \tilde{b} and \tilde{c} of \tilde{F} such that $i(\tilde{F}; \tilde{a}, \tilde{c}) - i(\tilde{F}; \tilde{b}, \tilde{c}) \neq 0$ which is independent of the choice of the isotopy I from Id_M to F (see [39, Section 4.2.4.2]). By Equation 1.2, we have $i(\tilde{F}^n; \tilde{a}, \tilde{b}) = n \cdot i(\tilde{F}; \tilde{a}, \tilde{b})$ for every $n \in \mathbb{N}$, where $i(\tilde{F}^n; \tilde{z}, \tilde{z}') = i_{\tilde{F}^n}(\tilde{z}, \tilde{z}')$ and \tilde{F}^n is the lifted identity isotopy of I^n to \tilde{M} . Hence we obtain that $|i(\tilde{F}^n; \tilde{a}, \tilde{c}) - i(\tilde{F}^n; \tilde{b}, \tilde{c})| \succeq n$. Note that the value of $\rho_{M,I}(\mu)$ only depend on the homotopic class with fixed endpoints of I . Therefore, if $\rho_{M,I}(\mu) \neq 0$, by the morphism property of $\rho_{M,\cdot}(\mu) : \mathcal{H}_*(M, \mu) \rightarrow H_1(M, \mathbb{R})$: $\rho_{M,II'}(\mu) = \rho_{M,I}(\mu) + \rho_{M,I'}(\mu)$ (see [10] for details), we have $\|\rho_{M,I^n}(\mu)\|_{H_1(M, \mathbb{R})} \succeq n$, where $\|\cdot\|_{H_1(M, \mathbb{R})}$ is a norm on the space $H_1(M, \mathbb{R})$.

Applying Proposition 1.8, Proposition 1.10, and the arguments above, we immediately obtain the following result:

Corollary 1.11. *The groups $\text{Hameo}(\mathbb{T}^2, \mu)$ and $\text{Homeo}_*(\Sigma_g, \mu)$ ($g > 1$) are torsion free.*

Note that it is easy to find a homeomorphism $F \in \text{Homeo}_*(\mathbb{T}^2, \mu) \setminus \text{Hameo}(\mathbb{T}^2, \mu)$ such that $F^n = \text{Id}_{\mathbb{T}^2}$ for $n > 1$, such as any rigidity rotation on \mathbb{T}^2 with rotation $\alpha \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$.

Let us finish this section by introduce the Zimmer's conjecture. Denote by $\text{Ham}^1(M, \mu)$ the group $\text{Hameo}(M, \mu) \cap \text{Diff}^1(M)$ and by $\text{Diff}_*^1(M, \mu)$ the group $\text{Homeo}_*(M, \mu) \cap \text{Diff}^1(M)$. Let us now recall the definition of distortion (see [32]). If \mathcal{G} is a finitely generated group with generators $\{g_1, \dots, g_s\}$, $f \in \mathcal{G}$ is a *distortion element* of \mathcal{G} provided that f has infinite order and

$$\liminf_{n \rightarrow +\infty} \frac{\|f^n\|_{\mathcal{G}}}{n} = 0,$$

where $\|f^n\|_{\mathcal{G}}$ is the word length of f^n in the generators $\{g_1, \dots, g_s\}$. If \mathcal{G} is not finitely generated, then we say that $f \in \mathcal{G}$ is distorted in \mathcal{G} if it is distorted in some finitely generated subgroup of \mathcal{G} .

Theorem 1.12. *Assume that $F \in \text{Diff}_*^1(\Sigma_g, \mu) \setminus \{\text{Id}_{\Sigma_g}\}$ ($g > 1$) (resp. $F \in \text{Ham}^1(\mathbb{T}^2, \mu) \setminus \{\text{Id}_{\mathbb{T}^2}\}$), and $\mathcal{G} \subset \text{Diff}_*^1(\Sigma_g, \mu)$ ($g > 1$) (resp. $\mathcal{G} \subset \text{Ham}^1(\mathbb{T}^2, \mu)$) is a finitely generated subgroup containing F , then*

$$\|F^n\|_{\mathcal{G}} \sim n.$$

As a consequence, the groups $\text{Diff}_^1(\Sigma_g, \mu)$ ($g > 1$) and $\text{Ham}^1(\mathbb{T}^2, \mu)$ have no distortion.*

Theorem 1.13. *Every homomorphism from $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) to $\text{Ham}^1(\mathbb{T}^2, \mu)$ or $\text{Diff}_*^1(\Sigma_g, \mu)$ ($g > 1$) is trivial. As a consequence, every homomorphism from $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) to $\text{Diff}_*^1(\Sigma_g, \mu)$ ($g > 1$) has only finite images.*

Theorem 1.13 is a more general conjecture of Zimmer [42] in the special surfaces case. Remark that Polterovich [32] showed us a Hamiltonian version of this theorem by using symplectic filling function, and that Franks and Handel [12] obtained this theorem by the Thurston theory of normal forms for surface homeomorphisms. Our strategy of proof is similar to the proof of Polterovich [32]. Hence our proof is totally different from Franks and Handel's. However, the technology of our proof is different from Polterovich's so that we can generalize his results to the group $\text{Ham}^1(\mathbb{T}^2, \mu)$ and the group $\text{Diff}_*^1(\Sigma_g, \mu)$ ($g > 1$), where μ is a usual Borel finite measure with full support. We note that the group $\text{Ham}^1(\mathbb{T}^2, \mu)$ is defined on the homology level (comparing to the definition of the classical Hamiltonian diffeomorphism which is defined on the co-homology level). The reader can find more information about Zimmer's conjecture in Section 7.

The article is organized as follows. In Section 2, we first introduce some notations, recall some results about identity isotopies, and study the WB-property on a connected subset of $\text{Fix}(\tilde{F})$. In Section 3, we explain the approach to defining the generalized action function and study the continuity of this action function. Our main results Proposition 1.4, Theorem 1.5 and Theorem 1.9 will be proved in Section 4, Section 5 and Section 6, respectively. In Section 7, we will provide an alternative proof of the C^1 -version of the Zimmer's conjecture on surfaces when the measure is a Borel finite measure with full support from our method. In Appendix, we provide the proofs of the lemmas which are not given in the main sections and also construct Example 8.3 and Example 8.4 to complete Theorem 1.5 and Theorem 1.6.

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2. NOTATIONS

We denote by $|\cdot|$ the usual Euclidean metric on \mathbb{R}^k or \mathbb{C}^k and by $\mathbf{S}^{k-1} = \{x \in \mathbb{R}^k \mid |x| = 1\}$ the unit sphere.

If A is a set, we write $\sharp A$ for the cardinality of A . If (S, σ, μ) is a measure space and V is any finite dimensional linear space, we denote by $L^1(S, V, \mu)$ the set of μ -integrable functions from S to V . If X is a topological space and A is a subset of X , we denote by $\text{Int}_X(A)$ and $\text{Cl}_X(A)$ respectively the interior and the closure of A . We will omit the subscript X if no confusion arises. If M is a manifold and N is a submanifold of M , we denote by ∂N the boundary of N on M .

2.1. Identity isotopies. An *identity isotopy* $I = (F_t)_{t \in [0,1]}$ on M is a continuous path

$$\begin{aligned} [0, 1] &\rightarrow \text{Homeo}(M) \\ t &\mapsto F_t \end{aligned}$$

such that $F_0 = \text{Id}_M$, where the last set is endowed with the compact-open topology. We naturally extend this map to \mathbb{R} by writing $F_{t+1} = F_t \circ F_1$. We can also define the inverse isotopy of I as $I^{-1} = (F_{-t})_{t \in [0,1]} = (F_{1-t} \circ F_1^{-1})_{t \in [0,1]}$. We denote by $\text{Homeo}_*(M)$ the set of all homeomorphisms of M that are isotopic to the identity.

A *path* on a manifold M is a continuous map $\gamma : J \rightarrow M$ defined on a nontrivial interval J (up to an increasing reparametrization). We can talk of a proper path (i.e. $\gamma^{-1}(K)$ is compact for any compact set K) or a compact path (i.e. J is compact). When γ is a compact path, $\gamma(\inf J)$ and $\gamma(\sup J)$ are the *ends* of γ . We say that a compact path γ is a *loop* if the two ends of γ coincide. The inverse of the path γ is defined by $\gamma^{-1} : t \mapsto \gamma(-t)$, $t \in -J$. If $\gamma_1 : J_1 \rightarrow M$ and $\gamma_2 : J_2 \rightarrow M$ are two paths such that

$$b_1 = \sup J_1 \in J_1, \quad a_2 = \inf J_2 \in J_2, \quad \text{and} \quad \gamma_1(b_1) = \gamma_2(a_2),$$

then the *concatenation* of γ_1 and γ_2 is defined on $J = J_1 \cup (J_2 + (b_1 - a_2))$ in the classical way, where $(J_2 + (b_1 - a_2))$ represents the translation of J_2 by $(b_1 - a_2)$:

$$\gamma_1 \gamma_2(t) = \begin{cases} \gamma_1(t) & \text{if } t \in J_1; \\ \gamma_2(t + a_2 - b_1) & \text{if } t \in J_2 + (b_1 - a_2). \end{cases}$$

Let \mathcal{I} be an interval (maybe infinite) of \mathbb{Z} . If $\{\gamma_i : J_i \rightarrow M\}_{i \in \mathcal{I}}$ is a family of compact paths satisfying that $\gamma_i(\sup(J_i)) = \gamma_{i+1}(\inf(J_{i+1}))$ for every $i \in \mathcal{I}$, then we can define their concatenation as $\prod_{i \in \mathcal{I}} \gamma_i$.

If $\{\gamma_i\}_{i \in \mathcal{I}}$ is a family of compact paths where $\mathcal{I} = \bigsqcup_{j \in \mathcal{J}} \mathcal{I}_j$ and \mathcal{I}_j is an interval of \mathbb{Z} such that $\prod_{i \in \mathcal{I}_j} \gamma_i$ is well defined (in the concatenation sense) for each $j \in \mathcal{J}$, we define their *product* by abusing notations:

$$\prod_{i \in \mathcal{I}} \gamma_i = \prod_{j \in \mathcal{J}} \prod_{i \in \mathcal{I}_j} \gamma_i.$$

The *trajectory* of a point z for the isotopy $I = (F_t)_{t \in [0,1]}$ is the oriented path $I(z) : t \mapsto F_t(z)$ defined on $[0, 1]$. Suppose that $\{I_k\}_{1 \leq k \leq k_0}$ is a family of identity isotopies on M . Write $I_k = (F_{k,t})_{t \in [0,1]}$. We can define a new identity isotopy $I_{k_0} \cdots I_2 I_1 = (F_t)_{t \in [0,1]}$ by concatenation as follows

$$(2.1) \quad F_t(z) = F_{k, k_0 t - (k-1)}(F_{k-1,1} \circ F_{k-2,1} \circ \cdots \circ F_{1,1}(z)) \quad \text{if} \quad \frac{k-1}{k_0} \leq t \leq \frac{k}{k_0}.$$

In particular, $I^{k_0}(z) = \prod_{k=0}^{k_0-1} I(F^k(z))$ when $I_k = I$ for all $1 \leq k \leq k_0$.

We write $\text{Fix}(F)$ for the set of fixed points of F . A fixed point z of $F = F_1$ is *contractible* if $I(z)$ is homotopic to zero. We write $\text{Fix}_{\text{Cont}, I}(F)$ for the set of contractible fixed points of F , which obviously depends on I .

2.2. The algebraic intersection number. Choosing an orientation on M permits us to define the algebraic intersection number $\Gamma \wedge \Gamma'$ between two loops. We keep the same notation $\Gamma \wedge \gamma$ for the algebraic intersection number between a loop Γ and a path γ when it is defined, e.g., when γ is proper or when γ is a compact path whose extremities are not in Γ . Similarly, we write $\gamma \wedge \gamma'$ for the algebraic intersection number of two paths γ and γ' when it is defined, e.g., when γ and γ' are compact paths and the ends of γ (resp. γ') are not on γ' (resp. γ). If Γ is a loop on a smooth manifold M , we write $[\Gamma] \in H_1(M, \mathbb{Z})$ for the homology class of Γ . It is clear that the value $\Gamma \wedge \gamma$ does not depend on the choice of the path γ whose endpoints are fixed when $[\Gamma] = 0$.

2.3. Rotation vector. Let us introduce the classical notion of rotation vector which was originally defined in [36]. Suppose that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$. Let $\text{Rec}^+(F)$ be the set of positively recurrent points of F . If $z \in \text{Rec}^+(F)$, we fix an open disk $U \subset M$ containing z , and write $\{F^{n_k}(z)\}_{k \geq 1}$ for the subsequence of the positive orbit of z obtained by keeping the points in U . For any $k \geq 0$, choose a simple path $\gamma_{F^{n_k}(z), z}$ in U joining $F^{n_k}(z)$ to z . The homology class $[\Gamma_k] \in H_1(M, \mathbb{Z})$ of the loop $\Gamma_k = I^{n_k}(z) \gamma_{F^{n_k}(z), z}$ is independent of the choice of $\gamma_{F^{n_k}(z), z}$. We say that z has a *rotation vector* $\rho_{M,I}(z) \in H_1(M, \mathbb{R})$ if

$$\lim_{l \rightarrow +\infty} \frac{1}{n_{k_l}} [\Gamma_{k_l}] = \rho_{M,I}(z)$$

for any subsequence $\{F^{n_{k_l}}(z)\}_{l \geq 1}$ which converges to z . Neither the existence nor the value of the rotation vector depends on the choice of U .

Suppose that M is compact and that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on M . Recall that $\mathcal{M}(F)$ is the set of Borel finite measures on M whose elements are invariant by F . If $\mu \in \mathcal{M}(F)$, we can define the rotation vector $\rho_{M,I}(z)$ for μ -almost every positively recurrent points. Moreover, we can prove that the rotation vector is uniformly bounded if it exists (see [39, page 52]). Therefore, we define the *rotation vector of the measure* to be

$$\rho_{M,I}(\mu) = \int_M \rho_{M,I} d\mu \in H_1(M, \mathbb{R}).$$

2.4. Some results about identity isotopies.

Remark 2.1. Suppose that M is an oriented compact surface and that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on M . When $z \in \text{Fix}_{\text{Cont}, I}(F)$, there is another identity isotopy $I' = (F'_t)_{t \in [0,1]}$ homotopic to I with fixed endpoints such that I' fixes z (see, e.g., [22, Proposition 2.15]). That is, there is a continuous map $H : [0, 1] \times [0, 1] \times M \rightarrow M$ such that

- $H(0, t, z) = F_t(z)$ and $H(1, t, z) = F'_t(z)$ for all $t \in [0, 1]$;
- $H(s, 0, z) = \text{Id}_M(z)$ and $H(s, 1, z) = F(z)$ for all $s \in [0, 1]$;
- $F'_t(z) = z$ for all $t \in [0, 1]$.

Lemma 2.2 ([39], page 54). *Let \mathbf{S}^2 be the 2-sphere and $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on \mathbf{S}^2 . For every three different fixed points z_i ($i = 1, 2, 3$) of F_1 , there exists another identity isotopy $I' = (F'_t)_{t \in [0,1]}$ from $\text{Id}_{\mathbf{S}^2}$ to F_1 such that I' fixes z_i ($i = 1, 2, 3$).*

As a consequence, we have the following corollary.

Corollary 2.3. *Let $I = (F_t)_{t \in [0,1]}$ be an identity isotopy on \mathbb{C} . For any two different fixed points z_1 and z_2 of F_1 , there exists another identity isotopy I' from $\text{Id}_{\mathbb{C}}$ to F_1 such that I' fixes z_1 and z_2 .*

Remark 2.4. Let $z_i \in \mathbf{S}^2$ ($i = 1, 2, 3$) and $\text{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)$ be the identity component of the space of all homeomorphisms of \mathbf{S}^2 leaving z_i ($i = 1, 2, 3$) pointwise fixed (for the compact-open topology). It is well known that $\pi_1(\text{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)) = 0$ [17, 18]. It implies that any two identity isotopies $I, I' \subset \text{Homeo}_*(\mathbf{S}^2, z_1, z_2, z_3)$ with fixed endpoints are homotopic. As a consequence, let $\text{Homeo}_*(\mathbb{C}, z_1, z_2)$ be the identity component of the space of all homeomorphisms of \mathbb{C} leaving two different points z_1 and z_2 of \mathbb{C} pointwise fixed, we have $\pi_1(\text{Homeo}_*(\mathbb{C}, z_1, z_2)) = 0$.

2.5. The linking number on a connected subset of the fixed points set. Let X be a connected component of $\text{Fix}_{\text{Cont}, I}(F)$. Either X is *contractible*, which means it is included in an open disk. In this case, the preimage of X in the universal covering space is a disjoint union of sets \tilde{X} such that the projection induces a homeomorphism from \tilde{X} to

X . Or X is not contractible, and in this case every connected component of the preimage of X is unbounded.

Recall the linking number $i(\tilde{F}; \tilde{z}, \tilde{z}')$ for $(\tilde{z}, \tilde{z}') \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ defined in Formula 1.2. To prove our main results, we need the following three lemmas whose proofs are provided in Appendix.

Lemma 2.5. *If \tilde{X} is a connected subset of $\text{Fix}(\tilde{F})$ and $\tilde{z} \in \text{Fix}(\tilde{F})$, then $i(\tilde{F}; \tilde{z}, \tilde{z}')$ (\tilde{z}' as variable, $\tilde{z}' \neq \tilde{z}$) is a constant on \tilde{X} . Furthermore, if \tilde{X} is not reduced to a singleton, $i(\tilde{F}; \cdot, \cdot)$ is a constant on $(\tilde{X} \times \tilde{X}) \setminus \tilde{\Delta}$.*

Lemma 2.6. *If \tilde{X} is a connected unbounded subset of $\text{Fix}(\tilde{F})$, then $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$ for all $(\tilde{z}, \tilde{z}') \in \text{Fix}(\tilde{F}) \times \tilde{X}$ with $\tilde{z} \neq \tilde{z}'$. Consequently, if X is a connected component of $\text{Fix}_{\text{Cont}, I}(F)$ and X is not contractible, $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$ for all $(\tilde{z}, \tilde{z}') \in \text{Fix}(\tilde{F}) \times \pi^{-1}(X)$ with $\tilde{z} \neq \tilde{z}'$.*

Lemma 2.7. *We have the following properties:*

- (1) *If X is a connected subset of $\text{Fix}_{\text{Cont}, I}(F)$ and X is not reduced to a singleton, I satisfies the B-property on $\pi^{-1}(X)$.*
- (2) *I satisfies the B-property if the number of the connected components of $\text{Fix}_{\text{Cont}, I}(F)$ is finite. In particular, I satisfies the B-property if the set $\text{Fix}_{\text{Cont}, I}(F)$ is connected.*

3. THE GENERALIZED ACTION FUNCTION REVISITED

In this section, we recall the approach to defining the generalized action function and show the continuity of this function.

3.1. The linking number of positively recurrent points.

Recall that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on a closed oriented surface M of genus $g \geq 1$ and that \tilde{F} is the time-one map of the lifted identity isotopy $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]}$ on the universal cover \tilde{M} of M . We can compactify \tilde{M} into a sphere by adding a point ∞ at infinity and then the lift \tilde{F} can be extended by fixing this point.

For every distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , by Lemma 2.2, we can choose an isotopy \tilde{I}_1 from $\text{Id}_{\tilde{M}}$ to \tilde{F} that fixes \tilde{a} and \tilde{b} . Recall that $\pi : \tilde{M} \rightarrow M$ is the covering map.

Fix $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$ and consider an open disk $U \subset M \setminus \pi(\{\tilde{a}, \tilde{b}\})$ containing z . We define the first return map $\Phi : \text{Rec}^+(F) \cap U \rightarrow \text{Rec}^+(F) \cap U$ and write $\Phi(z) = F^{\tau(z)}(z)$, where $\tau(z)$ is the first return time, that is, the least number $n \geq 1$ such that $F^n(z) \in U$. For every pair $(z', z'') \in U^2$, we choose an oriented simple path $\gamma_{z', z''}$ in U from z' to z'' . Denote by $\tilde{\Phi}$ the lift of the first return map Φ :

$$\begin{aligned} \tilde{\Phi} : \pi^{-1}(\text{Rec}^+(F)) \cap \pi^{-1}(U) &\rightarrow \pi^{-1}(\text{Rec}^+(F)) \cap \pi^{-1}(U) \\ \tilde{z} &\mapsto \tilde{F}^{\tau(\pi(\tilde{z}))}(\tilde{z}). \end{aligned}$$

For any $\tilde{z} \in \pi^{-1}(U)$, we write $U_{\tilde{z}}$ the connected component of $\pi^{-1}(U)$ that contains \tilde{z} . For each $j \geq 1$, let $\tau_j(z) = \sum_{i=0}^{j-1} \tau(\Phi^i(z))$. For every $n \geq 1$, consider the following curves in \tilde{M} :

$$\tilde{\Gamma}_{I_1, \tilde{z}}^n = \tilde{I}_1^{\tau_n(z)}(\tilde{z}) \tilde{\gamma}_{\tilde{\Phi}^n(\tilde{z}), \tilde{z}_n},$$

where $\tilde{z}_n \in \pi^{-1}(\{z\}) \cap \tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$ and $\tilde{\gamma}_{\tilde{\Phi}^n(\tilde{z}), \tilde{z}_n}$ is the lift of $\gamma_{\Phi^n(z), z}$ that is contained in $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})}$. We define the following infinite product (see Section 2.1):

$$\tilde{\Gamma}_{I_1, z}^n = \prod_{\pi(\tilde{z})=z} \tilde{\Gamma}_{I_1, \tilde{z}}^n.$$

In particular, when $z \in \text{Fix}(F)$, $\tilde{\Gamma}_{\tilde{I}_1, z}^1 = \prod_{\pi(\tilde{z})=z} \tilde{I}_1(\tilde{z})$.

When $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})} = \tilde{U}_{\tilde{z}}$, the curve $\tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n$ is a loop and hence $\tilde{\Gamma}_{\tilde{I}_1, z}^n$ is an infinite family of loops, which will be called a *multi-loop*. When $\tilde{U}_{\tilde{\Phi}^n(\tilde{z})} \neq \tilde{U}_{\tilde{z}}$, the curve $\tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n$ is a compact path and $\tilde{\Gamma}_{\tilde{I}_1, z}^n$ is therefore an infinite family of paths (it can be seen as a family of proper paths, which both ends of these paths go to ∞), which will be called a *multi-path*.

In the both cases, for every neighborhood \tilde{V} of ∞ , there must be finitely many loops or paths $\tilde{\Gamma}_{\tilde{I}_1, \tilde{z}}^n$ that are not included in \tilde{V} . By adding the point ∞ at infinity, we get a multi-loop on the sphere $\mathbf{S} = \tilde{M} \sqcup \{\infty\}$.

Hence $\tilde{\Gamma}_{\tilde{I}_1, z}^n$ can be seen as a multi-loop in the annulus $A_{\tilde{a}, \tilde{b}} = \mathbf{S} \setminus \{\tilde{a}, \tilde{b}\}$ with a finite homology. As a consequence, if $\tilde{\gamma}$ is a path from \tilde{a} to \tilde{b} , then the intersection number $\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n$ is well defined and does not depend on $\tilde{\gamma}$. By Remark 2.4 and the properties of intersection number, the intersection number depends on U but not on the choice of the identity isotopy \tilde{I}_1 . Moreover, by observing that the path $(\prod_{i=0}^{n-1} \gamma_{\Phi^{n-i}(z), \Phi^{n-i-1}(z)})(\gamma_{\Phi^n(z), z})^{-1}$ is a loop in U , we have

$$(3.1) \quad \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n = \tilde{\gamma} \wedge \prod_{j=0}^{n-1} \tilde{\Gamma}_{\tilde{I}_1, \Phi^j(z)}^1 = \sum_{j=0}^{n-1} \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, \Phi^j(z)}^1.$$

For $n \geq 1$, we can define the function

$$(3.2) \quad L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}_1, z}^n = \sum_{j=0}^{n-1} L_1(\tilde{F}; \tilde{a}, \tilde{b}, \Phi^j(z)),$$

where $U \subset M \setminus \pi(\{\tilde{a}, \tilde{b}\})$. The last equality of Equation 3.2 follows from Equation 3.1. And the function L_n depends on U but not on the choice of $\gamma_{\Phi^n(z), z}$.

Definition 3.1. Fix $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$. We say that the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z) \in \mathbb{R}$ is defined, if

$$\lim_{k \rightarrow +\infty} \frac{L_{n_k}(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau_{n_k}(z)} = i(\tilde{F}; \tilde{a}, \tilde{b}, z)$$

for any subsequence $\{\Phi^{n_k}(z)\}_{k \geq 1}$ of $\{\Phi^n(z)\}_{n \geq 1}$ which converges to z .

Note here that the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ does not depend on U since if U and U' are open disks containing z , there exists a disk containing z that is contained in $U \cap U'$. In particular, when $z \in \text{Fix}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ always exists and is equal to $L_1(\tilde{F}; \tilde{a}, \tilde{b}, z)$. Moreover, if $z \in \text{Fix}_{\text{Cont}, I}(F)$, we have [39, page 57]

$$(3.3) \quad i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_{\pi(\tilde{z})=z} \left(i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}) \right).$$

3.2. Some elementary properties of the linking number [39, Section 4.5.2].

Proposition 3.2. Let G be the covering transformation group of $\pi : \tilde{M} \rightarrow M$. For every $\alpha \in G$, all distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , and every $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$, we have $L_n(\tilde{F}; \alpha(\tilde{a}), \alpha(\tilde{b}), z) = L_n(\tilde{F}; \tilde{a}, \tilde{b}, z)$ for each n . If $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ exists, then $i(\tilde{F}; \alpha(\tilde{a}), \alpha(\tilde{b}), z)$ also exists and $i(\tilde{F}; \alpha(\tilde{a}), \alpha(\tilde{b}), z) = i(\tilde{F}; \tilde{a}, \tilde{b}, z)$.

Proposition 3.3. *For all distinct fixed points \tilde{a} , \tilde{b} and \tilde{c} of \tilde{F} , and every $z \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}, \tilde{c}\})$, we have $L_n(\tilde{F}; \tilde{a}, \tilde{b}, z) + L_n(\tilde{F}; \tilde{b}, \tilde{c}, z) + L_n(\tilde{F}; \tilde{c}, \tilde{a}, z) = 0$ for each n . Moreover, if two among the three linking numbers $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$, $i(\tilde{F}; \tilde{b}, \tilde{c}, z)$ and $i(\tilde{F}; \tilde{c}, \tilde{a}, z)$ exist, then the third one also exists and we have*

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) + i(\tilde{F}; \tilde{b}, \tilde{c}, z) + i(\tilde{F}; \tilde{c}, \tilde{a}, z) = 0.$$

The following lemma gives the continuity property of the function L_k .

Lemma 3.4. *Suppose that $\tilde{a} \in \text{Fix}(\tilde{F})$ and $\{\tilde{a}_n\}_{n \geq 1} \subset \text{Fix}(\tilde{F}) \setminus \{\tilde{a}\}$ satisfying $\tilde{a}_n \rightarrow \tilde{a}$ as $n \rightarrow +\infty$. Then we have*

- $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$ for $z \in \text{Fix}(F) \setminus \{\pi(\tilde{a})\}$;
- $\lim_{n \rightarrow +\infty} L_k(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$ for every $k \geq 1$ and $z \in \text{Rec}^+(F) \cap U$, where U is an open disk of $M \setminus \{\pi(\tilde{a})\}$.

Proof. Let $C_z^k = \pi^{-1}(\{z, F(z), \dots, F^{\tau_k(z)-1}(z)\})$, where $\tau_k(z) = \sum_{i=0}^{k-1} \tau(\Phi^i(z))$.

For each n , let \tilde{I}_n be the isotopy that fixes \tilde{a} , \tilde{a}_n and ∞ , as constructed in Lemma 2.2. Up to conjugacy by a homeomorphism $h : \tilde{M} \rightarrow \mathbb{C}$, we can identify \tilde{M} with the complex plane \mathbb{C} (refer to Remark 2.4 and Section 3.1 for the reasons). Through a simple computation (see the proof of Lemma 2.2 [39, page 54]), we can get the formula of \tilde{I}_n as follows

$$(3.4) \quad \tilde{I}_n(\tilde{z})(t) = \frac{\tilde{a}_n - \tilde{a}}{\tilde{F}_t(\tilde{a}_n) - \tilde{F}_t(\tilde{a})} \cdot (\tilde{F}_t(\tilde{z}) - \tilde{F}_t(\tilde{a})) + \tilde{a}.$$

Let \tilde{V}_n be a disk whose center is \tilde{a} and whose radius is $2|\tilde{a}_n - \tilde{a}|$. As the functions $L_k(\tilde{F}; \tilde{a}_n, \tilde{a}, z)$ do not depend on the path from \tilde{a} to \tilde{a}_n (see Section 3.1), we can suppose that the path $\tilde{\gamma}$ from \tilde{a}_n to \tilde{a} is always in \tilde{V}_n . Since $z \neq \pi(\tilde{a})$, the value

$$(3.5) \quad c = \liminf_{n \geq 1} \min_{t \in [0,1], \tilde{z} \in C_z^k} |\tilde{F}_t(\tilde{z}) - \tilde{F}_t(\tilde{a}_n)|$$

is positive and merely depends on z and k . For the constant c , we can find $N > 0$ large enough such that $\max_{t \in [0,1]} |\tilde{F}_t(\tilde{a}_n) - \tilde{F}_t(\tilde{a})| < c/3$ when $n \geq N$. This implies that, for every $\tilde{z} \in C_z^k$, $t \in [0,1]$, and $n \geq N$,

$$|\tilde{I}_n(\tilde{z})(t) - \tilde{a}| > 2|\tilde{a}_n - \tilde{a}|.$$

As a consequence, we have

$$\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0, \quad \text{for } z \in \text{Fix}(F) \setminus \{\pi(\tilde{a})\}$$

and

$$\lim_{n \rightarrow +\infty} L_k(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0, \quad \text{for } z \in \text{Rec}^+(F) \cap U.$$

□

3.3. Definition of the generalized action function.

Recall that F is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on M . In [39, page 85], we have proved that the function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is μ -integrable in each of the following cases:

- (1) the map F and its inverse F^{-1} are differentiable at $\pi(\tilde{a})$ and $\pi(\tilde{b})$;
- (2) the isotopy I satisfies the WB-property at \tilde{a} and \tilde{b} and the measure μ has full support;
- (3) the isotopy I satisfies the WB-property at \tilde{a} and \tilde{b} and the measure μ is ergodic.

Suppose now the function $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is μ -integrable. We define the *action difference* of \tilde{a} and \tilde{b} as follows

$$(3.6) \quad i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = \int_{M \setminus \pi(\{\tilde{a}, \tilde{b}\})} i(\tilde{F}; \tilde{a}, \tilde{b}, z) d\mu.$$

As an immediate consequence of Proposition 3.2, we have:

Corollary 3.5. $i_\mu(\tilde{F}; \alpha(\tilde{a}), \alpha(\tilde{b})) = i_\mu(\tilde{F}; \tilde{a}, \tilde{b})$ for any $\alpha \in G$.

We suppose now that the action difference $i_\mu(\tilde{F}; \tilde{a}, \tilde{b})$ is well defined for arbitrary two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} . We define the *action difference* as follows:

$$\begin{aligned} i_\mu : (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta} &\rightarrow \mathbb{R} \\ (\tilde{a}, \tilde{b}) &\mapsto i_\mu(\tilde{F}; \tilde{a}, \tilde{b}). \end{aligned}$$

Note that for each of the following cases, the action difference can be defined [39, page 86] for every pair $(\tilde{a}, \tilde{b}) \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$:

- $F \in \text{Diff}(M)$;
- I satisfies the WB-property and μ has full support;
- I satisfies the WB-property and μ is ergodic.

The following corollary is an immediate conclusion of Proposition 3.3:

Corollary 3.6. For any distinct fixed points \tilde{a} , \tilde{b} and \tilde{c} of \tilde{F} , we have

$$i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) + i_\mu(\tilde{F}; \tilde{b}, \tilde{c}) + i_\mu(\tilde{F}; \tilde{c}, \tilde{a}) = 0.$$

That is, i_μ is a coboundary on $\text{Fix}(\tilde{F})$. So there is a function $l_\mu : \text{Fix}(\tilde{F}) \rightarrow \mathbb{R}$, defined up to an additive constant, such that

$$i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = l_\mu(\tilde{F}; \tilde{b}) - l_\mu(\tilde{F}; \tilde{a}).$$

We call the function l_μ the *action function* on $\text{Fix}(\tilde{F})$ deduced by the measure μ .

From Corollary 3.5 and Corollary 3.6, we have the following proposition:

Proposition 3.7 ([39], page 87). If $\rho_{M,I}(\mu) = 0$, then $i_\mu(\tilde{F}; \tilde{a}, \alpha(\tilde{a})) = 0$ for each $\tilde{a} \in \text{Fix}(\tilde{F})$ and each $\alpha \in G \setminus \{e\}$, where e is the unit element of G . As a consequence, there exists a function L_μ defined on $\text{Fix}_{\text{Cont}, I}(F)$ such that for every two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , we have

$$i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = L_\mu(\tilde{F}; \pi(\tilde{b})) - L_\mu(\tilde{F}; \pi(\tilde{a})).$$

We call the function L_μ the *action function* on $\text{Fix}_{\text{Cont}, I}(F)$ defined by the measure μ . We proved that the function L_μ is a generalization of the classical case (Theorem 1.2, [39, Theorem 4.3.2]). By the construction of i_μ , the following property holds:

Proposition 3.8. The action difference i_μ (hence the action l_μ) only depends on the homotopic class with fixed endpoints of I . Moreover, i_μ only depends on the time-one map F when $g > 1$ and i_μ depends on the homotopic class of I when $g = 1$. The same property holds for L_μ (hence L_μ) which defines in the case where $\rho_{M,I}(\mu) = 0$.

3.4. The continuity of the generalized action function.

We have the following continuity property of the generalized action function whose proof details will be also used in the proof of Theorem 1.5.

Proposition 3.9. *Suppose that F is the time-one map of an identity isotopy I on M and that $\mu \in \mathcal{M}(F)$. Let $\tilde{X} \subseteq \text{Fix}(\tilde{F})$. If I satisfies the B-property on \tilde{X} and F is μ -symplectic, then we have*

$$\lim_{n \rightarrow +\infty} i_\mu(\tilde{F}; \tilde{a}_n, \tilde{a}) = 0$$

for any $\tilde{a} \in \tilde{X}$ and $\{\tilde{a}_n\}_{n \geq 1} \subset \tilde{X} \setminus \{\tilde{a}\}$ satisfying $\tilde{a}_n \rightarrow \tilde{a}$ as $n \rightarrow +\infty$ ². As a conclusion, if I satisfies the B-property on \tilde{X} and the WB-property (on $\text{Fix}(\tilde{F})$), the action l_μ is continuous on \tilde{X} . Moreover, if I is μ -Hamiltonian, the action function L_μ is continuous on $\pi(\tilde{X})$.

Proof. There exists a triangulation $\{U_i\}_{i=1}^{+\infty}$ of $M \setminus \text{Fix}(F)$ such that, for each i , the interior of U_i is an open free disk for F (i.e., $F(\text{Int}(U_i)) \cap \text{Int}(U_i) = \emptyset$) and satisfies $\mu(\partial U_i) = 0$. By a slight abuse of notations we will also write U_i for its interior.

According to Lemma 3.4, we have that $\lim_{n \rightarrow +\infty} i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$ for $z \in \text{Fix}(F) \setminus \{\pi(\tilde{a})\}$, and that $\lim_{n \rightarrow +\infty} L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z) = 0$ for $z \in \text{Rec}^+(F) \cap U_i$, for every $i \in \mathbb{N}$.

Choose a compact set $\tilde{P} \subset \tilde{M}$ such that $\tilde{a} \in \text{Int}(\tilde{P})$ and $\{\tilde{a}_n\}_{n \geq 1} \subset \tilde{P}$. As before, when \tilde{a}' and \tilde{b}' are two distinct fixed points of \tilde{F} in \tilde{P} , we can always suppose that the path $\tilde{\gamma}$ that joins \tilde{a}' and \tilde{b}' is in \tilde{P} . By the definition of B-property, we may suppose that there exists a number $N > 0$ such that

$$N > \text{ess sup}_{n \geq 1} \left\{ \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| \right\},$$

where “ess sup” is the essential supremum (see Proposition 4.6.11 in [39, page 85]).

By Lebesgue’s dominating convergence theorem (the dominated function is N), we get

$$\lim_{n \rightarrow +\infty} \int_{\text{Fix}(F)} \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu = 0.$$

It is then sufficient to prove that

$$\lim_{n \rightarrow +\infty} \int_{M \setminus \text{Fix}(F)} \left| i(\tilde{F}; \tilde{a}_n, \tilde{a}, z) \right| d\mu = 0.$$

Fix $\epsilon > 0$. Since $\mu(\bigcup_{i=1}^{+\infty} U_i) = \mu(M \setminus \text{Fix}(F)) < +\infty$, there exists a positive integer N' such that

$$\mu\left(\bigcup_{N'+1}^{+\infty} U_i\right) < \frac{\epsilon}{2N}.$$

For every pair $(\tilde{a}, \tilde{b}) \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$, each i , and μ -a.e. $z \in U_i$, we have the following facts:

- $\int_{U_i} \tau d\mu = \mu(\bigcup_{k \geq 0} F^k(U_i))$ (by Kac Lemma [23], see [39, page 55]);
- $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ is the action of F , i.e., $i(\tilde{F}; \tilde{a}, \tilde{b}, F(z)) = i(\tilde{F}; \tilde{a}, \tilde{b}, z)$. Indeed, one can consider the point $F(z) \in \text{Rec}^+(F)$ and the open disk $F(U_i)$. Then this fact follows from Definition 3.1.

Therefore,

$$(3.7) \quad \int_{\bigcup_{k \geq 0} F^k(U_i)} \left| i(\tilde{F}; \tilde{a}, \tilde{b}, z) \right| d\mu = \int_{U_i} \tau(z) \left| i(\tilde{F}; \tilde{a}, \tilde{b}, z) \right| d\mu.$$

²In fact, it is also true for the following cases (refer the proof of Proposition 6.8 in arXiv:1106.1104):

- (1) I satisfies the B-property on \tilde{X} and $F \in \text{Diff}(M)$;
- (2) I satisfies the B-property on \tilde{X} and μ is ergodic.

As $L_1(\tilde{F}; \tilde{a}, \tilde{b}, z) \in L^1(U_i, \mathbb{R}, \mu)$ (refer to Proposition 4.6.10 in [39, page 84] for the proof), the following limit exists

$$L^*(\tilde{F}; \tilde{a}, \tilde{b}, z) = \lim_{m \rightarrow +\infty} \frac{L_m(\tilde{F}; \tilde{a}, \tilde{b}, z)}{m} = \lim_{m \rightarrow +\infty} \frac{1}{m} \sum_{j=1}^{m-1} L_1(\tilde{F}; \tilde{a}, \tilde{b}, \Phi^j(z)).$$

Moreover, we have the following inequality (modulo subsets of measure zero of U_i)

$$\begin{aligned} (3.8) \quad \left| L^*(\tilde{F}; \tilde{a}, \tilde{b}, z) \right| &= \lim_{m \rightarrow +\infty} \frac{1}{m} \left| \sum_{j=0}^{m-1} (L_1(\tilde{F}; \tilde{a}, \tilde{b}, \Phi^j(z))) \right| \\ &\leq \lim_{m \rightarrow +\infty} \frac{1}{m} \sum_{j=0}^{m-1} \left| L_1(\tilde{F}; \tilde{a}, \tilde{b}, \Phi^j(z)) \right| \\ &= \left| L_1(\tilde{F}; \tilde{a}, \tilde{b}, z) \right|^*, \end{aligned}$$

where the last limit exists due to Birkhoff Ergodic theorem.

Applying Birkhoff Ergodic theorem again, we get

$$\tau^*(\Phi(z)) = \tau^*(z) \quad \text{and} \quad L^*(\tilde{F}; \tilde{a}, \tilde{b}, \Phi(z)) = L^*(\tilde{F}; \tilde{a}, \tilde{b}, z),$$

where $\tau^*(z)$ is the limit of the sequence $\{\tau_n(z)/n\}_{n \geq 1}$, and Φ is the first return map on U_i . For μ -a.e. $z \in U_i$, we have

$$(3.9) \quad i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \lim_{m \rightarrow +\infty} \frac{L_m(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau_m(z)} = \lim_{m \rightarrow +\infty} \frac{L_m(\tilde{F}; \tilde{a}, \tilde{b}, z)/m}{\tau_m(z)/m} = \frac{L^*(\tilde{F}; \tilde{a}, \tilde{b}, z)}{\tau^*(z)}.$$

Therefore, $i(\tilde{F}; \tilde{a}, \tilde{b}, \Phi(z)) = i(\tilde{F}; \tilde{a}, \tilde{b}, z)$. Moreover, observing that $\tau(z)|i(\tilde{F}; \tilde{a}, \tilde{b}, z)| \in L^1(U_i, \mathbb{R}, \mu)$, we obtain

$$\begin{aligned} &\lim_{m \rightarrow +\infty} \frac{1}{m} \sum_{j=0}^{m-1} \left(\tau(\Phi^j(z)) \left| i(\tilde{F}; \tilde{a}, \tilde{b}, \Phi^j(z)) \right| \right) \\ &= \lim_{m \rightarrow +\infty} \left(\frac{1}{m} \sum_{j=0}^{m-1} \tau(\Phi^j(z)) \right) \cdot \left| i(\tilde{F}; \tilde{a}, \tilde{b}, z) \right| \\ &= \tau^*(z) \left| i(\tilde{F}; \tilde{a}, \tilde{b}, z) \right| \end{aligned}$$

for μ -a.e. $z \in U_i$. This implies that

$$(3.10) \quad \int_{U_i} \tau(z) \left| i(\tilde{F}; \tilde{a}, \tilde{b}, z) \right| d\mu = \int_{U_i} \tau^*(z) \left| i(\tilde{F}; \tilde{a}, \tilde{b}, z) \right| d\mu.$$

From the equalities 3.7, 3.9, 3.10 and the inequality 3.8 above, we obtain

$$\begin{aligned}
\int_{\bigcup_{i=1}^{N'} U_i} |i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu &\leq \sum_{i=1}^{N'} \int_{\bigcup_{k \geq 0} F^k(U_i)} |i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \tau(z) |i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} \tau^*(z) |i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} |L^*(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu \\
&\leq \sum_{i=1}^{N'} \int_{U_i} |L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z)|^* \, d\mu \\
&= \sum_{i=1}^{N'} \int_{U_i} |L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu.
\end{aligned}$$

As N' is finite, according to Lebesgue's dominating convergence theorem (with the dominated function $N\tau(z)$) and Lemma 3.4, we have

$$\lim_{n \rightarrow +\infty} \sum_{i=1}^{N'} \int_{U_i} |L_1(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu = 0.$$

Therefore, there exists a positive number N'' such that when $n \geq N''$,

$$\int_{\bigcup_{i=1}^{N'} U_i} |i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu < \frac{\epsilon}{2}.$$

Finally, when $n \geq N''$, we obtain

$$\begin{aligned}
&\int_{M \setminus \text{Fix}(F)} |(i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu \\
&= \int_{\bigcup_{i=1}^{N'} U_i} |(i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu + \int_{\bigcup_{N'+1}^{\infty} U_i} |(i(\tilde{F}; \tilde{a}_n, \tilde{a}, z)| \, d\mu \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2N} \cdot N \\
&= \epsilon.
\end{aligned}$$

Hence, the first statement holds.

If $\rho_{M,I}(\mu) = 0$, to see L_μ is continuous on $\pi(\tilde{X})$, let $a \in \pi(\tilde{X})$ and let $\{a_n\}_{n \geq 1} \subset \pi(\tilde{X}) \setminus \{a\}$ converge to a . By Proposition 3.7, we only need to consider a lift $\tilde{a} \in \tilde{X}$ of a and a lifted sequence $\{\tilde{a}_n\}_{n \geq 1} \subset \tilde{X}$ of $\{a_n\}_{n \geq 1}$ that converges to \tilde{a} . Then it follows from the statement above. \square

4. THE PROOF OF PROPOSITION 1.4

Suppose that $X \subseteq \text{Fix}_{\text{Cont}, I}(F)$ is connected and not reduced to a singleton. By Lemma 2.7, I satisfies the B-property on $\pi^{-1}(X)$. If I satisfies the hypotheses of Theorem 1.2, according to Proposition 3.9, the action function L_μ is continuous on X . In fact, we have the following stronger result:

Proposition 1.4 *Under the hypotheses of Theorem 1.2, for every two distinct contractible fixed points a and b of F which belong to a same connected component of $\text{Fix}_{\text{Cont},I}(F)$, we have $I_\mu(\tilde{F}; a, b) = 0$. As a conclusion, the action function L_μ is a constant on each connected component of $\text{Fix}_{\text{Cont},I}(F)$.*

Given $Y \subset M$ and $\epsilon > 0$, let $Y_\epsilon = \{z \in M \mid d(z, y) < \epsilon, y \in Y\}$ be the ϵ -neighborhood of Y . If N is a submanifold of M , the inclusion $i : N \hookrightarrow M$ naturally induces a homomorphism: $i_* : \pi_1(N, p) \rightarrow \pi_1(M, p)$, where $p \in N$. To prove Proposition 1.4, we need the following topological lemma that will be proved in Appendix. The reader may find a similar version of Alexander-Spanier (co-)homology of this lemma (see [35]). The author thanks Le Calvez for the proof.

Lemma 4.1. *If Z is a connected compact subset of M and $z \in Z$, then there is $\epsilon_0 > 0$ such that*

$$i_*(\pi_1(Z_\epsilon, z)) = i_*(\pi_1(Z_{\epsilon_0}, z)) \quad \text{for all } 0 < \epsilon < \epsilon_0.$$

Proof of Proposition 1.4. Let X be a connected component of $\text{Fix}_{\text{Cont},I}(F)$ that is not a singleton. We will first consider the linking number $i(\tilde{F}; \tilde{a}, \tilde{b}, z)$ in the case where $z \in \text{Rec}^+(F) \setminus \text{Fix}_{\text{Cont},I}(F)$ and $(\tilde{a}, \tilde{b}) \in (\pi^{-1}(X) \times \pi^{-1}(X)) \setminus \tilde{\Delta}$.

Recall the following functions defined in Section 3.1:

$$L_k : ((\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}) \times (\text{Rec}^+(F) \cap U) \rightarrow \mathbb{Z},$$

$$L_k(\tilde{F}; \tilde{c}_1, \tilde{c}_2, z) = \tilde{\gamma} \wedge \tilde{\Gamma}_{I_1, z}^k = \sum_{j=0}^{k-1} L_1(\tilde{F}; \tilde{c}_1, \tilde{c}_2, \Phi^j(z)),$$

where $z \in \text{Rec}^+(F)$, $U \subset M \setminus \pi(\{\tilde{c}_1, \tilde{c}_2\})$ is an open disk containing z , and \tilde{I}_1 is an isotopy from $\text{Id}_{\tilde{M}}$ to \tilde{F} that fixes \tilde{c}_1 and \tilde{c}_2 .

We claim that, for every $z \in \text{Rec}^+(F) \setminus \text{Fix}_{\text{Cont},I}(F)$ and $k \geq 1$, there exists $\epsilon > 0$ which merely depends on z and k such that $L_k(\tilde{F}; \tilde{a}, \tilde{b}, z) = 0$ when $\tilde{d}(\tilde{a}, \tilde{b}) < \epsilon$.

Indeed, since X is compact, $z \notin X$, and $\tilde{F}_t \circ T = T \circ \tilde{F}_t$ for any $T \in G$, the value

$$c' = \min_{t \in [0,1], \tilde{z} \in C_z^k, \tilde{z}' \in \pi^{-1}(X)} |\tilde{F}_t(\tilde{z}) - \tilde{F}_t(\tilde{z}')|$$

is positive and only depends on z and k , where $C_z^k = \pi^{-1}(\{z, F(z), \dots, F^{\tau_k(z)-1}(z)\})$.

Recall that the isotopy

$$\tilde{I}'(\tilde{z})(t) = \frac{\tilde{b} - \tilde{a}}{\tilde{F}_t(\tilde{b}) - \tilde{F}_t(\tilde{a})} \cdot (\tilde{F}_t(\tilde{z}) - \tilde{F}_t(\tilde{a})) + \tilde{a}$$

fixes \tilde{a} , \tilde{b} and ∞ . Let $\epsilon > 0$ be small enough such that $\max_{t \in [0,1]} |\tilde{F}_t(\tilde{a}) - \tilde{F}_t(\tilde{b})| < c'/3$ when $\tilde{d}(\tilde{a}, \tilde{b}) < \epsilon$, and let \tilde{V}' be a disk whose center is \tilde{a} and radius is $2|\tilde{b} - \tilde{a}|$. The claim follows from the proof of Lemma 3.4 if one replaces \tilde{I}_n in Formula 3.4 by \tilde{I}' , \tilde{V}_n by \tilde{V}' , and c in Formula 3.5 by c' .

Fix $x \in X$ and a lift $\tilde{x} \in \tilde{M}$ of x . By Lemma 4.1, there is $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$,

$$i_*(\pi_1(X_\epsilon, x)) = i_*(\pi_1(X_{\epsilon_0}, x)).$$

Suppose that \tilde{X}_ϵ is the connected component of $\pi^{-1}(X_\epsilon)$ that contains \tilde{x} . Let $G_{\tilde{X}_\epsilon}$ be the stabilizer of \tilde{X}_ϵ in the group G , i.e., $G_{\tilde{X}_\epsilon} = \{T \in G \mid T(\tilde{X}_\epsilon) = \tilde{X}_\epsilon\}$. It is clear that $i_*(\pi_1(X_\epsilon, x)) \simeq G_{\tilde{X}_\epsilon}$. Hence $G_{\tilde{X}_{\epsilon_1}} = G_{\tilde{X}_{\epsilon_2}}$ for all $0 < \epsilon_2 < \epsilon_1 \leq \epsilon_0$. Let $\tilde{Y}_\epsilon = \tilde{X}_\epsilon \cap \pi^{-1}(X)$. Recall that X is connected. We have $\pi(\tilde{Y}_\epsilon) = X$ for all $0 < \epsilon \leq \epsilon_0$ since X_ϵ is path connected. Note that \tilde{Y}_ϵ is 4ϵ -chain connected, i.e., for any $\tilde{y}, \tilde{y}' \in \tilde{Y}_\epsilon$ there exists a sequence $\{\tilde{y}_i\}_{i=1}^n \subset \tilde{Y}_\epsilon$ such that $\tilde{y}_1 = \tilde{y}$, $\tilde{y}_n = \tilde{y}'$, and $\tilde{d}(\tilde{y}_i, \tilde{y}_{i+1}) < 4\epsilon$. Indeed, we can find

a path γ in X_ϵ from $\pi(\tilde{y})$ to $\pi(\tilde{y}')$ and a lift $\tilde{\gamma}$ of γ in \tilde{X}_ϵ from \tilde{y} to \tilde{y}' . On the path $\tilde{\gamma}$, we choose a sequence $\{\tilde{x}_i\}_{i=1}^n \subset \tilde{\gamma}$ such that $\tilde{x}_1 = \tilde{y}$, $\tilde{x}_n = \tilde{y}'$, and the disks $\{\tilde{D}(\tilde{x}_i, \epsilon)\}_{i=1}^n$ cover $\tilde{\gamma}$ with $\tilde{D}(\tilde{x}_i, \epsilon) \cap \tilde{D}(\tilde{x}_{i+1}, \epsilon) \neq \emptyset$ for all $i = 1, \dots, n-1$, where $\tilde{D}(\tilde{x}_i, \epsilon)$ is a disk on \tilde{M} whose center is \tilde{x}_i and radius is ϵ . Choose a sequence $\{\tilde{y}_i\}_{i=1}^n \subset \tilde{Y}_\epsilon$ such that $\tilde{y}_1 = \tilde{y}$, $\tilde{y}_n = \tilde{y}'$, and $\tilde{y}_i \in \tilde{D}(\tilde{x}_i, \epsilon) \cap \tilde{Y}_\epsilon$ for $2 \leq i \leq n-1$. Obviously, $\{\tilde{y}_i\}_{i=1}^n$ is a 4ϵ -chain in \tilde{Y}_ϵ from \tilde{y} to \tilde{y}' by the triangle inequality.

For any $y \in X$, we claim that $\tilde{y} \in \tilde{Y}_\epsilon$ for all $\tilde{y} \in \pi^{-1}(y) \cap \tilde{Y}_{\epsilon_0}$ and all $0 < \epsilon \leq \epsilon_0$. Otherwise, there is $0 < \epsilon_1 < \epsilon_0$ and $\tilde{y} \in \tilde{Y}_{\epsilon_0} \subset \tilde{X}_{\epsilon_0}$ such that $\tilde{y} \notin \tilde{Y}_{\epsilon_1}$, and hence $\tilde{y} \notin \tilde{X}_{\epsilon_1}$. However, there is a lift \tilde{y}' of y such that $\tilde{y}' \in \tilde{Y}_{\epsilon_1} \subset \tilde{X}_{\epsilon_1} \subset \tilde{X}_{\epsilon_0}$. On the one hand, $T \in G_{\tilde{X}_{\epsilon_0}}$ since $\tilde{y}, \tilde{y}' \in \tilde{X}_{\epsilon_0}$, where $\tilde{y} = T(\tilde{y}')$. On the other hand, $T \notin G_{\tilde{X}_{\epsilon_1}}$ since $\tilde{y} \notin \tilde{X}_{\epsilon_1}$. This is impossible because $G_{\tilde{X}_{\epsilon_1}} = G_{\tilde{X}_{\epsilon_0}}$ and hence the claim holds. This implies that $\tilde{Y}_\epsilon = \tilde{Y}_{\epsilon_0}$ for all $0 < \epsilon < \epsilon_0$, and thereby \tilde{Y}_{ϵ_0} is ϵ -chain connected for all $0 < \epsilon \leq \epsilon_0/4$.

Recall the equality in Proposition 3.3 for any distinct points \tilde{c}_1, \tilde{c}_2 and \tilde{c}_3 of $\text{Fix}(\tilde{F})$:

$$(4.1) \quad L_k(\tilde{F}; \tilde{c}_1, \tilde{c}_2, z) + L_k(\tilde{F}; \tilde{c}_2, \tilde{c}_3, z) + L_k(\tilde{F}; \tilde{c}_3, \tilde{c}_1, z) = 0.$$

Applying Equality 4.1, we get that, for all distinct points $\tilde{a}, \tilde{b} \in \tilde{Y}_{\epsilon_0}$, $L_k(\tilde{F}; \tilde{a}, \tilde{b}, z) = 0$ for all k and $z \in \text{Rec}^+(F) \setminus \text{Fix}_{\text{Cont}, I}(F)$. This implies that $i(\tilde{F}; \tilde{a}, \tilde{b}, z) = 0$ for all $(\tilde{a}, \tilde{b}) \in (\tilde{Y}_{\epsilon_0} \times \tilde{Y}_{\epsilon_0}) \setminus \tilde{\Delta}$ and $z \in \text{Rec}^+(F) \setminus \text{Fix}_{\text{Cont}, I}(F)$.

Let us now consider the case $z \in \text{Fix}_{\text{Cont}, I}(F)$ to finish our proof, which is in turn divided into two cases:

- (1) there is a set \tilde{X} on \tilde{M} which is a connected component of $\pi^{-1}(X)$ and satisfies that the covering map $\pi : \tilde{X} \rightarrow X$ is surjective (this case contains the case where X is path connected);
- (2) there is no such set satisfying Item 1.

Recall the linking number of z for \tilde{a} and \tilde{b} (see Formula 3.3):

$$i(\tilde{F}; \tilde{a}, \tilde{b}, z) = \sum_{\pi(\tilde{z})=z} \left(i(\tilde{F}; \tilde{a}, \tilde{z}) - i(\tilde{F}; \tilde{b}, \tilde{z}) \right),$$

where $(\tilde{a}, \tilde{b}) \in (\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ and $i(\tilde{F}; \tilde{c}, \tilde{z}) = i_{\tilde{F}}(\tilde{c}, \tilde{z})$ (see Formula 1.2).

In the first case, for any $\tilde{z} \in \pi^{-1}(z)$, by Lemma 2.5, $i(\tilde{F}; \tilde{z}', \tilde{z})$ is a constant (which depends on \tilde{z}) for all $\tilde{z}' \in \tilde{X} \setminus \{\tilde{z}\}$. We get that $i(\tilde{F}; \tilde{a}, \tilde{b}, z) = 0$ for any $(\tilde{a}, \tilde{b}) \in (\tilde{X} \times \tilde{X}) \setminus \tilde{\Delta}$ and $z \in \text{Fix}_{\text{Cont}, I}(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$.

Note that $\tilde{Y}_{\epsilon_0} = \tilde{X}$ in this case. Therefore, by the definition of the action function, we get that $i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = 0$ for all $(\tilde{a}, \tilde{b}) \in (\tilde{X} \times \tilde{X}) \setminus \tilde{\Delta}$. The conclusion follows from the fact that $\pi(\tilde{X}) = X$ and the hypothesis that $\rho_{M, I}(\mu) = 0$ in this case.

In the second case, write $\pi^{-1}(X)$ as $\bigsqcup_{\alpha \in \tilde{\Lambda}} \tilde{X}_\alpha$ where \tilde{X}_α is a connected component of $\pi^{-1}(X)$ on \tilde{M} . Note that $2 \leq \#\tilde{\Lambda} \leq +\infty$. It is easy to see that every such \tilde{X}_α is unbounded on \tilde{M} by the hypotheses and the connectedness of X .

Similar to the proof of the first case, for every $\alpha \in \tilde{\Lambda}$ and $\tilde{c} \in \tilde{X}_\beta$ with $\alpha \neq \beta$, the following property holds: when $z \in \text{Fix}_{\text{Cont}, I}(F)$, the linking number $i(\tilde{F}; \cdot, \tilde{c}, z) \in \mathbb{Z}$ is a constant on \tilde{X}_α , and hence $i(\tilde{F}; \tilde{a}, \tilde{b}, z) = 0$ for all $(\tilde{a}, \tilde{b}) \in (\tilde{X}_\alpha \times \tilde{X}_\alpha) \setminus \tilde{\Delta}$. Observing that every \tilde{X}_α is unbounded, we have that the constant is zero by Formula 3.3 and Lemma 2.6. Therefore, $i(\tilde{F}; \tilde{a}, \tilde{b}, z) = 0$ for all $(\tilde{a}, \tilde{b}) \in (\pi^{-1}(X) \times \pi^{-1}(X)) \setminus \tilde{\Delta}$.

Finally, by the definition of the action function, we get that $i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = 0$ for all $(\tilde{a}, \tilde{b}) \in (\tilde{Y}_{\epsilon_0} \times \tilde{Y}_{\epsilon_0}) \setminus \tilde{\Delta}$. The conclusion then follows from the facts that $\pi(\tilde{Y}_{\epsilon_0}) = X$ and that $\rho_{M, I}(\mu) = 0$ in the second case. \square

5. THE PROOF OF THEOREM 1.5

To prove Theorem 1.5, we need the following theorem:

Theorem 5.1 ([24, 28]). *Let M be a closed oriented surface with genus $g \geq 1$. If F is the time-one map of a μ -Hamiltonian isotopy I on M , then there exist at least three contractible fixed points of F .*

Remark that Theorem 5.1 is not valid when the measure has no full support (see Example 8.3 and Example 8.4 below).

Theorem 1.5 *Let F be the time-one map of a μ -Hamiltonian isotopy I on a closed oriented surface M with $g \geq 1$. If I satisfies the WB-property and F is not Id_M , the action function L_μ is not constant.*

Theorem 1.5 is proved in two cases: the set $\text{Fix}_{\text{Cont}, I}(F)$ is finite or infinite.

Proof of Theorem 1.5 for the case $\sharp \text{Fix}_{\text{Cont}, I}(F) < +\infty$.

We say that $X \subseteq \text{Fix}_{\text{Cont}, I}(F)$ is *unlinked* if there exists an isotopy $I' = (F'_t)_{t \in [0,1]}$ homotopic to I which fixes every point of X . Moreover, we say that X is a *maximal unlinked set* if any set $X' \subseteq \text{Fix}_{\text{Cont}, I}(F)$ that strictly contains X is not unlinked.

In the proof of Theorem 5.1 ([24, Theorem 10.1]), Le Calvez has proved that there exists a maximal unlinked set $X \subseteq \text{Fix}_{\text{Cont}, I}(F)$ with $\sharp X \geq 3$ if $\sharp \text{Fix}_{\text{Cont}, I}(F) < +\infty$.

There exists an oriented topological foliation \mathcal{F} on $M \setminus X$ (or, equivalently, a singular oriented foliation \mathcal{F} on M with X equal to the singular set) such that, for all $z \in M \setminus X$, the trajectory $I(z)$ is homotopic to an arc γ joining z and $F(z)$ in $M \setminus X$, which is positively transverse to \mathcal{F} . It means that for every $t_0 \in [0, 1]$ there exists an open neighborhood $V \subset M \setminus X$ of $\gamma(t_0)$ and an orientation preserving homeomorphism $h : V \rightarrow (-1, 1)^2$ which sends the foliation \mathcal{F} on the horizontal foliation (oriented with x_1 increasing) such that the map $t \mapsto p_2(h(\gamma(t)))$ defined in a neighborhood of t_0 is strictly increasing, where $p_2(x_1, x_2) = x_2$.

We can choose a point $z \in \text{Rec}^+(F) \setminus \text{Fix}(F)$ and a leaf λ containing z . Proposition 10.4 in [24] states that the ω -limit set $\omega(\lambda) \in X$, the α -limit set $\alpha(\lambda) \in X$, and $\omega(\lambda) \neq \alpha(\lambda)$. Fix an isotopy I' homotopic to I that fixes $\omega(\lambda)$ and $\alpha(\lambda)$ and a lift $\tilde{\lambda}$ of λ that joins $\widetilde{\omega(\lambda)}$ and $\widetilde{\alpha(\lambda)}$. Let us now study the linking number $i(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z')$ for $z' \in \text{Rec}^+(F) \setminus X$ when it exists. Observing that for all $z' \in M \setminus X$, the trajectory $I'(z')$ is still homotopic to an arc that is positively transverse to \mathcal{F} . Hence, for all $z' \in \text{Rec}^+(F) \setminus X$, without loss of generality, we can choose an open disk U containing z' such that $U \cap \lambda = \emptyset$ by shrinking U and perturbing λ if necessary. Then we get

$$L_n(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z') = \tilde{\lambda} \wedge \tilde{\Gamma}_{\tilde{I}', z'}^n = \lambda \wedge \Gamma_{I', z'}^n \geq 0$$

for every $n \geq 1$, where \tilde{I}' is the lift of I' to \tilde{M} and $\Gamma_{I', z'}^n = \pi(\tilde{\Gamma}_{\tilde{I}', z'}^n)$.

According to Definition 3.1, we have

$$i(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z') \geq 0$$

for μ -a.e. $z' \in \text{Rec}^+(F) \setminus \{\omega(\lambda), \alpha(\lambda)\}$.

By the continuity of I' and the hypothesis on μ , there exists an open free disk U containing z such that $\mu(U) > 0$ and $L_1(\tilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z') > 0$ when $z' \in U \cap \text{Rec}^+(F)$.

Similarly to the proof of Proposition 3.9, we obtain

$$\begin{aligned}
I_\mu(\widetilde{F}; \omega(\lambda), \alpha(\lambda)) &\geq \int_{\bigcup_{k \geq 0} F^k(U)} i(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\
&= \int_U \tau(z) i(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\
&= \int_U \tau^*(z) i(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\
&= \int_U L^*(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\
&= \int_U L_1(\widetilde{F}; \widetilde{\omega(\lambda)}, \widetilde{\alpha(\lambda)}, z) d\mu \\
&> 0.
\end{aligned}$$

□

Before proving the case where the set $\text{Fix}_{\text{Cont}, I}(F)$ is infinite, let us recall two results:

Proposition 5.2 (Franks' Lemma [9]). *Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an orientation preserving homeomorphism. If F possesses a periodic free disk chain, a family $(U_r)_{r \in \mathbb{Z}/n\mathbb{Z}}$ of pairwise disjoint free topological open disks, such that for every $r \in \mathbb{Z}/n\mathbb{Z}$, one of the positive iterates of U_r meets U_{r+1} , then F has at least one fixed point.*

Theorem 5.3 ([22]). *Let M be an oriented surface and F be the time-one map of an identity isotopy I on M . There exists a closed subset $X \subset \text{Fix}(F)$ and an isotopy I' in $\text{Homeo}(M \setminus X)$ joining $\text{Id}_{M \setminus X}$ to $F|_{M \setminus X}$ such that*

- (1) *For all $z \in X$, the loop $I(z)$ is homotopic to zero in M .*
- (2) *For all $z \in \text{Fix}(F) \setminus X$, the loop $I'(z)$ is not homotopic to zero in $M \setminus X$.*
- (3) *For all $z \in M \setminus X$, the trajectories $I(z)$ and $I'(z)$ are homotopic with fixed endpoints in M .*
- (4) *There exists an oriented topological foliation \mathcal{F} on $M \setminus X$ such that, for all $z \in M \setminus X$, the trajectory $I'(z)$ is homotopic to an arc γ joining z and $F(z)$ in $M \setminus X$ which is positively to \mathcal{F} .*

Moreover, the isotopy I' satisfies the following property:

- (5) *For all finite $Y \subset X$, there exists an isotopy I'_Y joining Id_M and F in $\text{Homeo}(M)$ which fixes Y such that, if $z \in M \setminus X$, the arc $I'(z)$ and $I'_Y(z)$ are homotopic in $M \setminus Y$. And if $z \in X \setminus Y$, the loop $I'_Y(z)$ is contractible in $M \setminus Y$.*

Proof of Theorem 1.5 for the case $\sharp \text{Fix}_{\text{Cont}, I}(F) = +\infty$.

Suppose that X , I' and \mathcal{F} are respectively the closed contractible fixed points set, the isotopy, and the foliation, as stated in Theorem 5.3. Obviously, $X \neq \emptyset$ (see Remark 2.1) and $\mu(M \setminus X) > 0$. Assume that X' is the union of the connected components of X that separate M . Write $M \setminus X' = \sqcup_i S_i$ where S_i is an open F -invariant connected subsurface of M (see [3]). For every i , we denote by I'_i the restriction of I' on $S_i \setminus (S_i \cap X)$. We claim that

$$\sharp \{\text{the connected components of } \partial S_i \cup (S_i \cap X)\} \geq 2 \quad \text{for every } i.$$

To prove this claim, we can suppose that $\sharp \{\text{the connected components of } \partial S_i \cup (X \cap S_i)\}$ is finite. If S_i is a subsurface of the sphere, we only need to consider the case of disk. In this case, by Proposition 5.2 and Item 2 of Theorem 5.3, $X \cap S_i \neq \emptyset$ and thereby the claim follows. When S_i is not a subsurface of the sphere, we can get a closed surface S'_i through compactifying S_i . More precisely, we add one point on each connected component of $\partial S_i \cup (S_i \cap X)$. Note that $S_i \setminus (S_i \cap X)$ is embedded in S'_i and we can extend I'_i on $S_i \setminus (S_i \cap X)$ to an identity isotopy on S'_i which is still denoted by I'_i . To be more precise,

let $X'_i = S'_i \setminus (S_i \setminus (S_i \cap X))$ be the added points set. Observing that X'_i is totally connected (in fact finite), the isotopy I'_i can be extended to an identity isotopy on S'_i that fixes every point in X'_i (see [22, Remark 1.18]). By the definitions of S'_i and I'_i , and the items 1 and 3 of Theorem 5.3, we get $\rho_{S'_i, I'_i}(\mu) = 0 \in H_1(S'_i, \mathbb{R})$. According to Item 2 of Theorem 5.3, X'_i is a maximal unlink set of I'_i . Thanks to the proof of Theorem 5.1 ([24, Theorem 10.1]), we have $\sharp X'_i \geq 3$. Therefore, the claim holds.

Fix one connected set S_i . Similarly to the finite case, we choose a point $z \in (\text{Rec}^+(F) \setminus \text{Fix}(F)) \cap S_i$ and a leaf $\lambda \in \mathcal{F}$ containing z . In [25], the proofs of Proposition 4.1 and 4.3 (page 150 and 152, for S_i being a subset of the sphere) and of Proposition 6.1 (page 166, for S_i being not a subset of the sphere) imply that the ω -set of λ , $\omega(\lambda)$ (resp. the α -set of λ , $\alpha(\lambda)$), is connected and is contained in a connected component of $\partial S_i \cup (X \cap S_i)$. We write the connected component as $X_+(\lambda)$ (resp. $X_-(\lambda)$). Moreover, $X_+(\lambda) \neq X_-(\lambda)$. Choose a lift $\tilde{\lambda}$ of λ . We need to consider the following four cases: the set $\omega(\tilde{\lambda})$ or $\alpha(\tilde{\lambda})$ contains ∞ or not.

Take two points $a \in \alpha(\lambda)$ and $b \in \omega(\lambda)$. Let $Y = \{a, b\}$ and I'_Y be the isotopy as in Theorem 5.3. Suppose that \tilde{I}'_Y is the identity lift of I'_Y to \tilde{M} . Notice that

- (A1): if $z \in M \setminus X$, then the arcs $I'(z)$ and $I'_Y(z)$ are homotopic in $M \setminus Y$ (Item 5, Theorem 5.3), and $I'_Y(z)$ is homotopic to an arc γ from z to $F(z)$ in $M \setminus Y$ and positively transverse to \mathcal{F} (Item 4, Theorem 5.3);
- (A2): if $z \in X \setminus Y$, then $\gamma \wedge I'_Y(z) = 0$ where γ is any path from a to b (Item 5, Theorem 5.3).

We now suppose that neither $\alpha(\tilde{\lambda})$ nor $\omega(\tilde{\lambda})$ contains ∞ . Replacing $\alpha(\lambda)$ by a , $\omega(\lambda)$ by b , and I' by I'_Y in the proof of the finite case, we can get $I_\mu(\tilde{F}; a, b) > 0$.

We now consider the case that either $\alpha(\tilde{\lambda})$ or $\omega(\tilde{\lambda})$ contains ∞ . Recall that \tilde{d} is the distance on \tilde{M} induced by a distance d on M which is further induced by a Riemannian metric on M . Define $\tilde{d}(\tilde{z}, \tilde{C}) = \inf_{\tilde{c} \in \tilde{C}} \tilde{d}(\tilde{z}, \tilde{c})$, if $\tilde{z} \in \tilde{M}$ and $\tilde{C} \subset \tilde{M}$. Take a sequence $\{(\tilde{a}_m, \tilde{b}_m)\}_{m \geq 1}$ such that

- $\pi(\tilde{a}_m) = a$ and $\pi(\tilde{b}_m) = b$;
- if $\alpha(\tilde{\lambda})$ (resp. $\omega(\tilde{\lambda})$) does not contain ∞ , we set $\tilde{a}_m = \tilde{a}$ (resp. $\tilde{b}_m = \tilde{b}$) for every m where $\tilde{a} \in \pi^{-1}(a) \cap \alpha(\tilde{\lambda})$ (resp. $\tilde{b} \in \pi^{-1}(b) \cap \omega(\tilde{\lambda})$);
- $\lim_{m \rightarrow +\infty} \tilde{d}(\tilde{a}_m, \tilde{\lambda}) = 0$ and $\lim_{m \rightarrow +\infty} \tilde{d}(\tilde{b}_m, \tilde{\lambda}) = 0$.

For every m , there exists \tilde{c}_m (resp. \tilde{c}'_m) on $\tilde{\lambda}$ such that $\tilde{d}(\tilde{a}_m, \tilde{c}_m) = \tilde{d}(\tilde{a}_m, \tilde{\lambda})$ (resp. $\tilde{d}(\tilde{b}_m, \tilde{c}'_m) = \tilde{d}(\tilde{b}_m, \tilde{\lambda})$). Note that $\tilde{c}_m = \tilde{a}_m = \tilde{a}$ (resp. $\tilde{c}'_m = \tilde{b}_m = \tilde{b}$) and $\tilde{d}(\tilde{a}_m, \tilde{\lambda}) = 0$ (resp. $\tilde{d}(\tilde{b}_m, \tilde{\lambda}) = 0$) if $\alpha(\tilde{\lambda})$ (resp. $\omega(\tilde{\lambda})$) does not contain ∞ . Choose a simple smooth path \tilde{l}_m (resp. \tilde{l}'_m) from \tilde{a}_m (resp. \tilde{c}'_m) to \tilde{c}_m (resp. \tilde{b}_m) such that the length of \tilde{l}_m (resp. \tilde{l}'_m) converges to zero as $m \rightarrow +\infty$ and $\pi(\tilde{l}_{m+1}) \subset \pi(\tilde{l}_m)$ (resp. $\pi(\tilde{l}'_{m+1}) \subset \pi(\tilde{l}'_m)$). Here, we assume that the simple path \tilde{l}_m (resp. \tilde{l}'_m) is empty if $\alpha(\tilde{\lambda})$ (resp. $\omega(\tilde{\lambda})$) does not contain ∞ . Let $\tilde{\gamma}_m = \tilde{l}_m \tilde{\lambda}_m \tilde{l}'_m$, where $\tilde{\lambda}_m$ is the sub-path of $\tilde{\lambda}$ from \tilde{c}_m to \tilde{c}'_m . Then $\tilde{\gamma}_m$ is a path from \tilde{a}_m to \tilde{b}_m .

We know that, for every $m \geq 1$, the linking number $i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z')$ exists for μ -a.e. $z' \in M \setminus \{a, b\}$. Hence, the linking number $i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z')$ exists on a full measure subset of $M \setminus \{a, b\}$ for all m .

According to A2 above, we have $i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') = 0$ if $z' \in X \setminus \{a, b\}$. To finish Theorem 1.5, we need the following lemma whose proof will be provided afterwards.

Lemma 5.4. $\liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') \geq 0$ for μ -a.e. $z' \in \text{Rec}^+(F) \setminus X$.

Armed with this lemma we obtain

$$(5.1) \quad \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') \geq 0 \quad \text{for } \mu\text{-a.e. } z' \in \text{Rec}^+(F) \setminus \{a, b\}.$$

From the continuity of I'_Y and the hypothesis on μ , there exists an open free disk U containing z such that $\mu(U) > 0$ and for $z' \in U \cap \text{Rec}^+(F)$,

$$(5.2) \quad \lim_{m \rightarrow +\infty} L_1(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') > 0.$$

As $\rho_{M,I}(\mu) = 0$, by Proposition 3.7, the inequalities 5.1 and 5.2, and Fatou's lemma, we have

$$\begin{aligned} I_\mu(\tilde{F}; a, b) &= \lim_{m \rightarrow +\infty} i_\mu(\tilde{F}; \tilde{a}_m, \tilde{b}_m) \\ &= \lim_{m \rightarrow +\infty} \int_{M \setminus \{a, b\}} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\ &\geq \int_{M \setminus \{a, b\}} \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\ &\geq \int_{\bigcup_{k \geq 0} F^k(U)} \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\ &= \int_U \liminf_{m \rightarrow +\infty} \tau(z) i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\ &= \int_U \liminf_{m \rightarrow +\infty} \tau^*(z) i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\ &= \int_U \liminf_{m \rightarrow +\infty} L^*(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\ &= \int_U \liminf_{m \rightarrow +\infty} L_1(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z) d\mu \\ &> 0. \end{aligned}$$

Therefore, we only need to prove Lemma 5.4.

Proof of Lemma 5.4. Fix one point $z' \in \text{Rec}^+(F) \setminus X$ and choose an open disk U containing z' (here again, we suppose that $U \cap \lambda = \emptyset$). By **A1** and the construction of $\tilde{\gamma}_m$, for every $n \geq 1$, there exists $m(z', n) \in \mathbb{N}$ such that when $m \geq m(z', n)$, the value

$$(5.3) \quad L_n(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') = \tilde{\gamma}_m \wedge \tilde{\Gamma}_{I'_Y, z'}^n = \pi(\tilde{\gamma}_m) \wedge \Gamma_{I'_Y, z'}^n \geq 0$$

is constant with regard to m .

We prove this lemma by contradiction. Suppose that

$$\mu\{z' \in \text{Rec}^+(F) \setminus X \mid \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') < 0\} > 0.$$

There exists a small number $c > 0$ such that

$$(5.4) \quad \mu\{z' \in \text{Rec}^+(F) \setminus X \mid \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') < -c\} > c.$$

Write $E = \{z' \in \text{Rec}^+(F) \setminus X \mid \liminf_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') < -c\}$. Fix a point $z' \in E$ and an open disk U containing z' as before. By taking a subsequence if necessary, we can suppose that

$$-\infty \leq \lim_{m \rightarrow +\infty} i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') < -c.$$

Then there exists $N(z')$ such that for $m \geq N(z')$,

$$i(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') = \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z')}{\tau_n(z')} < -c.$$

Fix $m_0 \geq N(z')$. There exists $n(z', m_0) \in \mathbb{N}$ such that when $n \geq n(z', m_0)$,

$$\frac{L_n(\tilde{F}; \tilde{a}_{m_0}, \tilde{b}_{m_0}, z')}{\tau_n(z')} < -c.$$

Then we can choose $n_0 \geq n(z', m_0)$ such that

$$L_{n_0}(\tilde{F}; \tilde{a}_{m_0}, \tilde{b}_{m_0}, z') < -c\tau_{n_0}(z').$$

Based on Inequality 5.3, there exists $m(z', n_0) > m_0$ such that when $m \geq m(z', n_0)$,

$$L_{n_0}(\tilde{F}; \tilde{a}_m, \tilde{b}_m, z') \geq 0.$$

By fixing $m_1 \geq m(z', n_0)$, there exists $n(z', m_1) > n_0$ such that when $n \geq n(z', m_1)$,

$$\frac{L_n(\tilde{F}; \tilde{a}_{m_1}, \tilde{b}_{m_1}, z')}{\tau_n(z')} < -c.$$

Then we can choose $n_1 \geq n(z', m_1)$ such that

$$L_{n_1}(\tilde{F}; \tilde{a}_{m_1}, \tilde{b}_{m_1}, z') < -c\tau_{n_1}(z').$$

By induction, we can construct a sequence $\{(m_i, n_i)\}_{i \geq 0} \subset \mathbb{N} \times \mathbb{N}$ satisfying that

(B1): $\{m_i\}_{i \geq 0}$ and $\{n_i\}_{i \geq 0}$ are strictly increasing sequences;

(B2): for all $i \geq 0$, we have

$$L_{n_i}(\tilde{F}; \tilde{a}_{m_i}, \tilde{b}_{m_i}, z') < -c\tau_{n_i}(z') \quad \text{and} \quad L_{n_i}(\tilde{F}; \tilde{a}_{m_{i+1}}, \tilde{b}_{m_{i+1}}, z') \geq 0.$$

According to the positively transverse property of \mathcal{F} , it is clear that the negative part of the value $L_{n_i}(\tilde{F}; \tilde{a}_{m_i}, \tilde{b}_{m_i}, z')$ comes from at least one of the algebraic intersection numbers of the curve $\tilde{\Gamma}_{\tilde{I}'_Y, z'}^n$ with \tilde{l}_{m_i} and \tilde{l}'_{m_i} .

We deal with the case where both $\alpha(\tilde{\lambda})$ and $\omega(\tilde{\lambda})$ contain ∞ , and other cases follow similarly. In this case, the both sets $\alpha(\lambda), \omega(\lambda) \subset X$ are not contractible. According to Item 5 of Theorem 5.3, for any $z'' \in M \setminus Y$, the loop $I_Y'^{-1}I'(z'')$ is contractible in $M \setminus Y$ (see Section 2.1 for the definition of $I_Y'^{-1}$). It implies that $\tilde{\gamma}_m \wedge I_Y'^{-1}I'(z'') = 0$ for all m and $z'' \in \text{Rec}^+(F) \setminus X$. Note that I_Y' fixes a and b , the loop $I_Y'(z'')$ is contractible in $M \setminus Y$ for any $z'' \in X \setminus Y$, but $\alpha(\lambda)$ and $\omega(\lambda)$ are not contractible. By the continuity of I_Y' , we get $|\pi(\tilde{l}_{m_i}) \wedge I_Y'(x)| \leq 1$ (resp. $|\pi(\tilde{l}'_{m_i}) \wedge I_Y'(x)| \leq 1$) if the algebraic intersection number is defined³ and x is close to a (resp. b). Based on the construction of $\tilde{\lambda}_m$, **B1** and **B2**, there must be a sequence of open disks $\{U_i^a\}_{i \geq 0}$ containing the set $(I_Y')^{-1}(\pi(\tilde{l}_{m_i})) = \cup_{y \in \pi(\tilde{l}_{m_i})} (I_Y')^{-1}(y)$ (resp. $\{U_i^b\}_{i \geq 0}$ containing the set $(I_Y')^{-1}(\pi(\tilde{l}'_{m_i})) = \cup_{y \in \pi(\tilde{l}'_{m_i})} (I_Y')^{-1}(y)$) that satisfies

(C1): $U_{i+1}^a \subset U_i^a$ (resp. $U_{i+1}^b \subset U_i^b$) and $\mu(U_i^a) \rightarrow 0$ (resp. $\mu(U_i^b) \rightarrow 0$) as $i \rightarrow +\infty$ (since the measure μ has no atoms on $\text{Fix}_{\text{Cont}, I}(F)$);

(C2): for every $i \geq 0$,

$$\frac{1}{\tau_{n_i}(z')} \sum_{j=0}^{\tau_{n_i}(z')-1} \chi_{U_i^a} \circ F^j(z') > \frac{c}{2} \quad \text{or} \quad \frac{1}{\tau_{n_i}(z')} \sum_{j=0}^{\tau_{n_i}(z')-1} \chi_{U_i^b} \circ F^j(z') > \frac{c}{2},$$

where χ_U is the indicator function of $U \subset M$.

³The algebraic intersection number of $\pi(\tilde{l}_{m_i}) \wedge I_Y'(x)$ (resp. $\pi(\tilde{l}'_{m_i}) \wedge I_Y'(x)$) is well defined for μ -a.e. $x \in \text{Rec}^+(F)$ if $\mu(\pi(\tilde{l}_{m_i})) = 0$ (resp. $\mu(\pi(\tilde{l}'_{m_i})) = 0$), which can be easily done by slightly perturbing λ and \tilde{l}_{m_i} (resp. \tilde{l}'_{m_i}) if necessary.

Denote by $\chi_U^*(x)$ the limit of $\frac{1}{n} \sum_{j=0}^{n-1} \chi_U \circ F^j(x)$ as $n \rightarrow +\infty$ for μ -a.e. $x \in M$ (due to Birkhoff Ergodic theorem). By **C2** and Inequality 5.4, for each i , we have

$$\mu(\{x \in \text{Rec}^+(F) \setminus X \mid \chi_{U_i^a}^*(x) \geq \frac{c}{2} \text{ or } \chi_{U_i^b}^*(x) \geq \frac{c}{2}\}) > c.$$

This implies that $\int_M (\chi_{U_i^a}^*(x) + \chi_{U_i^b}^*(x)) d\mu \geq \frac{c^2}{2} > 0$ for every i . On the other hand, thanks to Birkhoff Ergodic theorem and **C1**, we have

$$\int_M (\chi_{U_i^a}^*(x) + \chi_{U_i^b}^*(x)) d\mu = \int_M (\chi_{U_i^a}(x) + \chi_{U_i^b}(x)) d\mu = \mu(U_i^a) + \mu(U_i^b) \rightarrow 0$$

as $i \rightarrow +\infty$, which gives a contradiction. We have finished the proof of Lemma 5.4. \square

We have completed the proof of Theorem 1.5. \square

6. THE PROOF OF THEOREM 1.9

Theorem 1.9 *Let F be the time-one map of an identity isotopy I on a closed oriented surface M with $g > 1$. If I satisfies the WB-property, $F \in \text{Homeo}(M) \setminus \{\text{Id}_M\}$, and $\mu \in \mathcal{M}(F)$ has full support, then there exist two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} such that $i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) \neq 0$.*

Proof. If $\rho_{M,I}(\mu) = 0$, by Theorem 1.5, there exist two distinct contractible fixed points a and b of F such that $I_\mu(\tilde{F}; a, b) \neq 0$, thus for any their lifts \tilde{a} and \tilde{b} we have $i_\mu(\tilde{F}; \tilde{a}, \tilde{b}) = I_\mu(\tilde{F}; a, b) \neq 0$.

If $\rho_{M,I}(\mu) \neq 0$, there exists $\alpha \in G$ such that $\varphi(\alpha) \wedge \rho_{M,I}(\mu) \neq 0$, where φ is the Hurewitz homomorphism from G to $H_1(M, \mathbb{Z})$. By Lefschetz-Nielsen's formula, we know that $\text{Fix}_{\text{Cont}, I}(F) \neq \emptyset$. Choose $a \in \text{Fix}_{\text{Cont}, I}(F)$ and a lift \tilde{a} of a . There exists an isotopy I' homotopic to I that fixes a (see Remark 2.1). It is lifted to an isotopy \tilde{I}' that fixes \tilde{a} and $\alpha(\tilde{a})$. Let $z \in \text{Rec}^+(F)$ and U be an open disk that contains z . Observe that if $\tilde{\gamma}$ is an oriented path from \tilde{a} to $\alpha(\tilde{a})$, then the intersection number $\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}', z}^n$ is equal to the intersection of the loop $\pi(\tilde{\gamma})$ with the loop $\Gamma_{I', z}^n = I'^{\tau_n(z)}(z) \gamma_{\Phi^n(z), z}$ (see Section 3.1). Recall the fact that

$$\lim_{n \rightarrow +\infty} \frac{[\Gamma_{I', z'}^n]}{\tau_n(z')} = \rho_{M, I'}(z')$$

for μ -a.e. $z' \in U$ (see [39, pages 54-56]). We get that

$$\begin{aligned} i_\mu(\tilde{F}; \tilde{a}, \alpha(\tilde{a})) &= \int_{M \setminus \{\pi(\tilde{a})\}} i(\tilde{F}; \tilde{a}, \alpha(\tilde{a}), z) d\mu \\ &= \int_{M \setminus \{\pi(\tilde{a})\}} \lim_{n \rightarrow +\infty} \frac{L_n(\tilde{F}; \tilde{a}, \alpha(\tilde{a}), z)}{\tau_n(z)} d\mu \\ &= \int_{M \setminus \{\pi(\tilde{a})\}} \lim_{n \rightarrow +\infty} \frac{\tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}', z}^n}{\tau_n(z)} d\mu \\ &= \pi(\tilde{\gamma}) \wedge \rho_{M, I'}(\mu) \\ &= \varphi(\alpha) \wedge \rho_{M, I}(\mu) \\ &\neq 0. \end{aligned}$$

We have finished the proof. \square

7. THE ABSENCE OF DISTORTION IN $\text{Ham}^1(\mathbb{T}^2, \mu)$ AND $\text{Diff}_*^1(\Sigma_g, \mu)$ WITH $g > 1$.

In 2002, Polterovich [32] showed us a Hamiltonian version of the Zimmer program (see [42]) dealing with actions of lattices. It is achieved by using the classical action defined in symplectic geometry, the symplectic filling function (see Section 1.2 in [32]), and Schwarz's theorem which we have mentioned in the beginning of this article. In 2003, Franks and Handel [11] developed the Thurston theory of normal forms for surface homeomorphisms with finite fixed sets. In 2006, they [12] used the generalized normal form to give a more general version (the map is a C^1 -diffeomorphism and the measure is a Borel finite measure) of the Zimmer program on the closed oriented surfaces. We recommend the reader a survey by Fisher [8] and an article by Brown, Fisher and Hurtado [2] for the recent progress of Zimmer program. We will give an alternative proof of the C^1 -version of the Zimmer's conjecture on surfaces when the measure is a Borel finite measure with full support.

Suppose that F is a C^1 -diffeomorphism of Σ_g ($g \geq 1$) which is the time-one map of an identity isotopy $I = (F_t)_{t \in [0,1]}$ on Σ_g and \tilde{F} is the time-one map of the lifted identity isotopy $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]}$ on the universal cover \tilde{M} of Σ_g . Recall that, if $\mathcal{G} \subset \text{Diff}_*^1(\Sigma_g, \mu)$ is a finitely generated subgroup containing F , $\|F\|_{\mathcal{G}}$ is the word length of F in \mathcal{G} . We have the following proposition whose proof will be provided in Appendix.

Proposition 7.1. *If there exist two distinct fixed points \tilde{a} and \tilde{b} of \tilde{F} , and a point $z_* \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$ such that $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ exists and is not zero, then for any finitely generated subgroup $F \in \mathcal{G} \subset \text{Diff}_*^1(\Sigma_g, \mu)$ ($g \geq 1$),*

$$\|F^n\|_{\mathcal{G}} \succeq \sqrt{n}.$$

If $F \neq \text{Id}_{\Sigma_g}$ and μ has full support, by Item **A2** in the proof of Theorem 1.5, we can choose $z_* \in \text{Rec}^+(F) \setminus X$, such that $\rho_{\Sigma_g, I}(z_*)$ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ exist, and $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ is not zero. By Proposition 7.1, we can get the following result which is a generalization of Theorem 1.6 B in [32] on the closed surfaces.

Corollary 7.2. *If $F \in \text{Diff}_*^1(\Sigma_g, \mu) \setminus \{\text{Id}_{\Sigma_g}\}$ ($g > 1$) or $F \in \text{Ham}^1(\mathbb{T}^2, \mu) \setminus \{\text{Id}_{\mathbb{T}^2}\}$, then for any finitely generated subgroup $F \in \mathcal{G} \subset \text{Diff}_*^1(\Sigma_g, \mu)$ ($g \geq 1$),*

$$\|F^n\|_{\mathcal{G}} \succeq \sqrt{n}.$$

Moreover, we can improve Corollary 7.2. The following result is our main theorem in this section.

Theorem 1.12 *Let $F \in \text{Diff}_*^1(\Sigma_g, \mu) \setminus \{\text{Id}_{\Sigma_g}\}$ ($g > 1$) (resp. $F \in \text{Ham}^1(\mathbb{T}^2, \mu) \setminus \{\text{Id}_{\mathbb{T}^2}\}$), and $\mathcal{G} \subset \text{Diff}_*^1(\Sigma_g, \mu)$ ($g > 1$) (resp. $\mathcal{G} \subset \text{Ham}^1(\mathbb{T}^2, \mu)$) be a finitely generated subgroup containing F , then*

$$\|F^n\|_{\mathcal{G}} \sim n.$$

As a consequence, the groups $\text{Diff}_^1(\Sigma_g, \mu)$ ($g > 1$) and $\text{Ham}^1(\mathbb{T}^2, \mu)$ have no distortions.*

Theorem 1.12 can be obtained immediately from the following two lemmas which will be proved in Appendix.

Lemma 7.3. *If $F \in \text{Homeo}_*(\Sigma_g, \mu) \setminus \text{Hameo}(\Sigma_g, \mu)$ ($g > 1$), for any finitely generated subgroup $F \in \mathcal{G} \subset \text{Homeo}_*(\Sigma_g, \mu)$, we have $\|F^n\|_{\mathcal{G}} \sim n$.*

Lemma 7.4. *If $F \in \text{Ham}^1(\Sigma_g, \mu) \setminus \{\text{Id}_{\Sigma_g}\}$ ($g \geq 1$), for any finitely generated subgroup $F \in \mathcal{G} \subset \text{Diff}_*^1(\Sigma_g, \mu)$, we have $\|F^n\|_{\mathcal{G}} \sim n$.*

As a consequence of Theorem 1.12, we have the following theorem:

Theorem 7.5. *Let \mathcal{G} be a finitely generated group with generators $\{g_1, \dots, g_s\}$ and $f \in \mathcal{G}$ be an element which is distorted with respect to the word norm on \mathcal{G} . Then $\phi(f) = \text{Id}_{\mathbb{T}^2}$ (resp. $\phi(f) = \text{Id}_{\Sigma_g}$ where $g > 1$) for any homomorphism $\phi : \mathcal{G} \rightarrow \text{Ham}^1(\mathbb{T}^2, \mu)$ (resp.*

$\phi : \mathcal{G} \rightarrow \text{Diff}_*^1(\Sigma_g, \mu)$ with $g > 1$). In particular, if \mathcal{G} is a finitely generated subgroup of $\text{Ham}^1(\mathbb{T}^2, \mu)$ (resp. $\text{Diff}_*^1(\Sigma_g, \mu)$ with $g > 1$), every element of $\mathcal{G} \setminus \{\text{Id}_{\Sigma_g}\}$ ($g \geq 1$) is undistorted with respect to the word norm on \mathcal{G} .

Proof. We only prove the case where $\phi : \mathcal{G} \rightarrow \text{Ham}^1(\mathbb{T}^2, \mu)$ since other cases follow similarly. Let \mathcal{G}' be the finitely generated group generated by $\{\phi(g_1), \dots, \phi(g_s)\}$. As f is a distortion element of \mathcal{G} , there exists a subsequence $\{n_i\}_{i \geq 1} \subset \mathbb{N}$ such that

$$\lim_{i \rightarrow +\infty} \frac{\|\phi^{n_i}(f)\|_{\mathcal{G}'}}{n_i} = \lim_{i \rightarrow +\infty} \frac{\|\phi(f^{n_i})\|_{\mathcal{G}'}}{n_i} = 0.$$

By Theorem 1.12, we have $\phi(f) = \text{Id}_{\mathbb{T}^2}$. \square

Let us recall some results about the irreducible lattice $\text{SL}(n, \mathbb{Z})$ with $n \geq 3$. The lattice $\text{SL}(n, \mathbb{Z})$ and its any normal subgroup of finite order have the following properties:

- It contains a subgroup isomorphic to the group of upper triangular integer valued matrices of order 3 with 1 on the diagonal (the integer Heisenberg group), which tells us the existence of distortion element of every infinite normal subgroup of $\text{SL}(n, \mathbb{Z})$ (see [33, Proposition 1.7]);
- It is almost simple (every normal subgroup is finite or has a finite index) which is due to Margulis (Margulis finiteness theorem, see [29]).

Applying these results above and Theorem 7.5, we get the following result:

Theorem 1.13 *Every homomorphism from $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) to $\text{Ham}^1(\mathbb{T}^2, \mu)$ or $\text{Diff}_*^1(\Sigma_g, \mu)$ ($g > 1$) is trivial. As a consequence, every homomorphism from $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) to $\text{Diff}^1(\Sigma_g, \mu)$ ($g > 1$) has only finite images.*

Proof. Again, we only prove the case where $\phi : \mathcal{G} \rightarrow \text{Ham}^1(\mathbb{T}^2, \mu)$ since other cases follow similarly. The following argument is due to Polterovich [33, Proof of Theorem 1.6]. By the first item of properties of $\text{SL}(n, \mathbb{Z})$, there is a distortion element f in $\text{SL}(n, \mathbb{Z})$. Apply Theorem 7.5 to the distortion element f of infinite order of $\text{SL}(n, \mathbb{Z})$. We have that f lies in the kernel of ϕ . Note that $\text{Ker}(\phi)$ is an infinite normal subgroup of $\text{SL}(n, \mathbb{Z})$. By the second item of properties of $\text{SL}(n, \mathbb{Z})$, $\text{Ker}(\phi)$ has finite index in $\text{SL}(n, \mathbb{Z})$. Hence the quotient $\text{SL}(n, \mathbb{Z})/\text{Ker}(\phi)$ is finite. Therefore, ϕ has finite images. Applying Corollary 1.11, we get ϕ is trivial.

Finally, let us recall a classical result about the mapping class group $\text{Mod}(S)$, where S is a compact, orientable, connected surface, possibly with boundary, and $\text{Mod}(S) = \text{Homeo}^+(S)/\text{Homeo}_*(S)$ is the isotopy classes of orientation preserving homeomorphisms of S (see [13]): any homomorphism $\phi : \Gamma \rightarrow \text{Mod}(S)$ has finite images where Γ is an irreducible lattice in a semisimple lie group of \mathbb{R} -rank at least two.

Applying the results above, we get the last statement: every homomorphism from $\text{SL}(n, \mathbb{Z})$ ($n \geq 3$) to $\text{Diff}^1(\Sigma_g, \mu)$ ($g > 1$) has only finite images, which is a general conjecture of Zimmer in the special case of surfaces. \square

8. APPENDIX

8.1. Proofs of Lemma 2.5, Lemma 2.6 and Lemma 2.7.

Endow the surface M with a Riemannian metric and denote by d the distance induced by the metric. Lift the Riemannian metric to \widetilde{M} and write \widetilde{d} for the distance induced by the metric. Let us recall some properties of $i(\widetilde{F}; \widetilde{z}, \widetilde{z}')$ defined in Formula 1.2, whose proofs can be found in [39, page 56]):

(P1): $i(\widetilde{F}; \widetilde{z}, \widetilde{z}')$ is locally constant on $(\text{Fix}(\widetilde{F}) \times \text{Fix}(\widetilde{F})) \setminus \widetilde{\Delta}$;

(P2): $i(\tilde{F}; \tilde{z}, \tilde{z}')$ is invariant by covering transformation, that is,

$$i(\tilde{F}; \alpha(\tilde{z}), \alpha(\tilde{z}')) = i(\tilde{F}; \tilde{z}, \tilde{z}') \quad \text{for every } \alpha \in G;$$

(P3): $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$ if $\pi(\tilde{z}) = \pi(\tilde{z}')$;

(P4): there exists K such that $i(\tilde{F}; \tilde{z}, \tilde{z}') = 0$ if $\tilde{d}(\tilde{z}, \tilde{z}') \geq K$.

Lemma 2.6 immediately follows from Lemma 2.5 and P4. Hence we only need to prove Lemma 2.5 and Lemma 2.7.

Proof of Lemma 2.5. If $\tilde{z} \notin \tilde{X}$, the conclusion holds obviously by the continuity (see P1 above) and the connectedness of \tilde{X} . Suppose now that $\tilde{z} \in \tilde{X}$. Fix a point $\tilde{a} \in \tilde{X}$. The linking number $i(\tilde{F}; \tilde{a}, \cdot)$ will be a constant on each connected component of $\tilde{X} \setminus \tilde{a}$. Let \tilde{b} and \tilde{b}' , \tilde{c} and \tilde{c}' lie on different components, respectively. Then $i(\tilde{F}; \tilde{a}, \tilde{b}) = i(\tilde{F}; \tilde{a}, \tilde{b}')$ and $i(\tilde{F}; \tilde{a}, \tilde{c}) = i(\tilde{F}; \tilde{a}, \tilde{c}')$. We have to prove that $i(\tilde{F}; \tilde{a}, \tilde{b}) = i(\tilde{F}; \tilde{a}, \tilde{c})$. We now fix \tilde{b} . Let \tilde{Y} be the connected component of $\tilde{X} \setminus \{\tilde{a}\}$ that contains \tilde{c} . Then \tilde{a} belongs to the closure of \tilde{Y} and hence $\tilde{Y} \cup \tilde{a}$ is connected. Let \tilde{Z} be the connected component of $\tilde{X} \setminus \{\tilde{b}\}$ that contains \tilde{a} . So $(\tilde{Y} \cup \tilde{a}) \cap \tilde{Z} \neq \emptyset$. Hence $\tilde{c} \in \tilde{Y} \subset \tilde{Z}$. We get $i(\tilde{F}; \tilde{b}, \tilde{a}) = i(\tilde{F}; \tilde{b}, \tilde{c})$ since \tilde{a} and \tilde{c} lie on the same connected component of $\tilde{X} \setminus \tilde{b}$. Now fix \tilde{c} . Similarly, we have $i(\tilde{F}; \tilde{c}, \tilde{b}) = i(\tilde{F}; \tilde{c}, \tilde{a})$ since \tilde{b} and \tilde{a} lie on the same connected component of $\tilde{X} \setminus \tilde{c}$. Obviously, $i(\tilde{F}; \tilde{z}, \tilde{z}')$ is symmetrical on $(\text{Fix}(\tilde{F}) \times \text{Fix}(\tilde{F})) \setminus \tilde{\Delta}$ by the definition of $i_{\tilde{F}}(\tilde{z}, \tilde{z}')$. Therefore, we obtain

$$i(\tilde{F}; \tilde{a}, \tilde{b}) = i(\tilde{F}; \tilde{b}, \tilde{a}) = i(\tilde{F}; \tilde{b}, \tilde{c}) = i(\tilde{F}; \tilde{c}, \tilde{b}) = i(\tilde{F}; \tilde{c}, \tilde{a}) = i(\tilde{F}; \tilde{a}, \tilde{c}).$$

□

We now show that Lemma 2.7 holds. Prior to that, we establish the following lemma:

Lemma 8.1. *If \tilde{X} is a connected subset of $\text{Fix}(\tilde{F})$ and \tilde{X} is not reduced to a singleton, I satisfies the B-property on \tilde{X} .*

Proof. Let \tilde{X}' be a connected component of $\text{Fix}(\tilde{F})$ that contains \tilde{X} . By Lemma 2.6, the conclusion is obvious if \tilde{X}' is unbounded. Suppose now that \tilde{X}' is bounded. Then \tilde{X}' is compact. Let us consider the value $i(\tilde{F}; \tilde{z}, \tilde{z}')$ where $\tilde{z} \in \text{Fix}(\tilde{F})$ and $\tilde{z}' \in \tilde{X}'$. By the second statement of Lemma 2.5, we only need to consider the case where $\tilde{z} \in \text{Fix}(\tilde{F}) \setminus \tilde{X}'$. If there exists a sequence $\{\tilde{z}_n\}_{n=1}^{+\infty} \subset \text{Fix}(\tilde{F}) \setminus \tilde{X}'$ such that $|i(\tilde{F}; \tilde{z}_n, \tilde{z}')| \rightarrow +\infty$ as $n \rightarrow +\infty$, according to P4, the sequence $\{\tilde{z}_n\}$ must have a convergence subsequence. Without loss of generality, we suppose that $\lim_{n \rightarrow +\infty} \tilde{z}_n = \tilde{z}_0$. Obviously, $\tilde{z}_0 \notin \tilde{X}'$ by the second statement of Lemma 2.5, which is also impossible in this case since $\tilde{d}(\tilde{z}_0, \tilde{X}') > 0$ and due to the first statement of Lemma 2.5. □

Proof of Lemma 2.7. If X is not contractible, the first property of Lemma 2.7 follows from Lemma 2.6. Otherwise, it follows from Lemma 8.1 and the properties P2–P4 of $i(\tilde{F}; \tilde{z}, \tilde{z}')$.

Furthermore, we assume that the number of connected components of $\text{Fix}_{\text{Cont}, I}(F)$ is finite. If some connected components of $\text{Fix}_{\text{Cont}, I}(F)$ are singletons, the second property follows from the properties P2–P4 and the first property of this lemma has been proved above. □

8.2. Proof of Lemma 4.1.

To prove Lemma 4.1, we need the following lemma:

Lemma 8.2. *Let S and S' be two open connected subsurfaces of an orientable closed surface M , with $S' \subset \text{Int}(S)$ and $z \in S'$. If S and S' satisfy the following properties:*

- $i_*(H_1(S, \mathbb{Z}))$ and $i_*(H_1(S', \mathbb{Z}))$ have the same image in $H_1(M, \mathbb{Z})$;

- the number of connected components with positive genus of $\text{Cl}(M \setminus S)$ equals to that (i.e., the number of connected components with positive genus) of $\text{Cl}(M \setminus S')$, then $i_*(\pi_1(S, z))$ and $i_*(\pi_1(S', z))$ have the same image in $\pi_1(M, z)$.

Proof. Let C be a connected component of $\partial S'$ which belongs to $\partial(\text{Cl}(S \setminus S'))$, more precisely, to the boundary of a connected component S'' of $\text{Cl}(S \setminus S')$. The genus of S'' is zero because $i_*(H_1(S, \mathbb{Z}))$ and $i_*(H_1(S', \mathbb{Z}))$ have the same image in $H_1(M, \mathbb{Z})$. We claim that C is the unique common component of $\partial S'$ and $\partial S''$. Otherwise, one can find a cycle r in S that has a nonzero algebraic intersection number with C , which contradicts with the fact that $i_*(H_1(S, \mathbb{Z}))$ and $i_*(H_1(S', \mathbb{Z}))$ have the same image in $H_1(M, \mathbb{Z})$ (see the left, Figure 1). Secondly, we note that $i_*(H_1(\text{Cl}(M \setminus S), \mathbb{Z}))$ and $i_*(H_1(\text{Cl}(M \setminus S'), \mathbb{Z}))$ have the same image in $H_1(M, \mathbb{Z})$ by the hypotheses. It implies that S'' (in fact, every connected component of $\text{Cl}(S \setminus S')$) is a subsurface without genus (see the right part of Figure 1 for a counter-example).

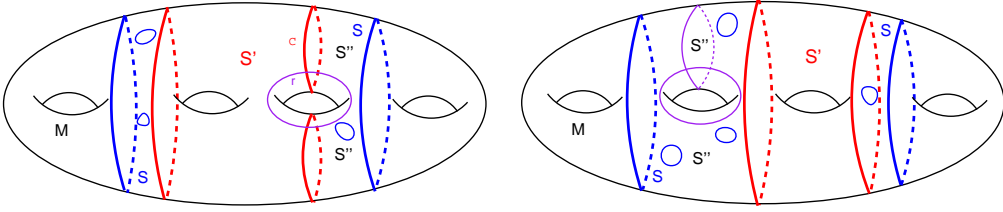


FIGURE 1.

Furthermore, by the hypotheses, we can deduce that S'' (and hence every connected component of $\text{Cl}(S \setminus S')$) satisfies the following property: the number of the boundary circles of S'' that bound a nonzero genus subsurface of $M \setminus S$ is at most 1 (see Figure 2, the left situation can happen but the right one can not). All the other boundary circles of S'' not bounding a nonzero genus subsurface of $M \setminus S$ will bound disks, say D_i ($1 \leq i \leq m$). And each D_i is contained in $M \setminus S$. It implies that every path in S'' whose endpoints are on C is homotopic in $S'' \cup (\cup_{i=1}^m D_i)$ to a path on C . Thus, the conclusion follows.

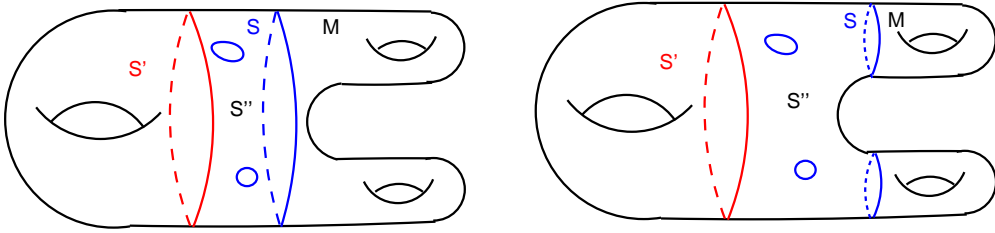


FIGURE 2.

□

Proof of Lemma 4.1. For a small $\epsilon > 0$, the inclusion $i : Z_\epsilon \hookrightarrow M$ naturally induces a homomorphism: $i_* : H_1(Z_\epsilon, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z})$. Observing that the sequences of subspaces $\{i_*(H_1(Z_\epsilon, \mathbb{Z}))\}_{\epsilon > 0}$ and $\{i_*(H_1(M \setminus Z_\epsilon, \mathbb{Z}))\}_{\epsilon > 0}$ are respectively non-decreasing and non-increasing in a finite dimensional space (since $H_1(M, \mathbb{Z})$ is finitely generated), we get that they must stabilize. Therefore, we can choose a $\epsilon_0 > 0$ small enough such that

$$i_*(H_1(Z_\epsilon, \mathbb{Z})) = i_*(H_1(Z_{\epsilon_0}, \mathbb{Z})) \quad \text{and} \quad i_*(H_1(\text{Cl}(M \setminus Z_\epsilon), \mathbb{Z})) = i_*(H_1(\text{Cl}(M \setminus Z_{\epsilon_0}), \mathbb{Z}))$$

for all $0 < \epsilon < \epsilon_0$. Furthermore, we can make ϵ_0 even smaller such that for all $0 < \epsilon < \epsilon_0$, the subsurfaces Z_{ϵ_0} and Z_ϵ satisfy the hypotheses of Lemma 8.2 since the genus of M is finite. The conclusion then follows from Lemma 8.2. \square

8.3. Proofs of Proposition 7.1, Lemma 7.3 and Lemma 7.4.

Proof of Proposition 7.1. If $z_* \in \text{Rec}^+(F) \setminus \pi(\{\tilde{a}, \tilde{b}\})$ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ exists, we have $z_* \in \text{Rec}^+(F^n) \setminus \pi(\{\tilde{a}, \tilde{b}\})$ (see Lemma 19, [41]), and by Proposition 4.5.3 in [39], we have that $i(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) = ni(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ for all $n \geq 1$.

Assume that $F \in \mathcal{G} = \langle F_{1,1}, \dots, F_{s,1} \rangle \subset \text{Diff}_*^1(\Sigma_g, \mu)$ and write $N(n) = \|F^n\|_{\mathcal{G}}$. Then there exist identity isotopies $I_i = (F_{i,t})_{t \in [0,1]} \subset \text{Diff}^1(\Sigma_g)$ ($1 \leq i \leq s$) such that, for every $n \geq 1$, the isotopy I^n is homotopic to the isotopy $I^{(n)} := (F_t^{(n)})_{0 \leq t \leq 1} = \prod_{j=1}^{N(n)} I_{i_j}^{\epsilon_j}$, where $i_j \in \{1, 2, \dots, s\}$, $\epsilon_j \in \{-1, 1\}$ ($j = 1, 2, \dots, N(n)$) and

$$F_t^{(n)}(z) = F_{i_k, N(n)t - (k-1)}^{\epsilon_{i_k}}(F_{i_{k-1}, 1}^{\epsilon_{i_{k-1}}} \circ \dots \circ F_{i_1, 1}^{\epsilon_{i_1}}(z)), \quad \text{if } \frac{k-1}{N(n)} \leq t \leq \frac{k}{N(n)}.$$

Let $\tilde{I}_i = (\tilde{F}_{i,t})_{t \in [0,1]}$ ($1 \leq i \leq s$) and $\tilde{I}^{(n)} = (\tilde{F}_t^{(n)})_{0 \leq t \leq 1}$ be the lifts of I_i ($1 \leq i \leq s$) and $(F_t^{(n)})_{0 \leq t \leq 1}$ to \tilde{M} respectively. Identify the sphere $\tilde{M} \cup \{\infty\}$ as the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Again, for simplicity, we can suppose that $\tilde{a} = 0$ and $\tilde{b} = 1$.

Fix $n \geq 1$. Using the method of Lemma 2.2 (see [39], page 57), we can get the isotopy $\tilde{I}'^{(n)} = (\tilde{F}_t'^{(n)})_{0 \leq t \leq 1}$ which fixes 0, 1 and is an isotopy on \tilde{M} from $\text{Id}_{\tilde{M}}$ to \tilde{F}^n , where

$$(8.1) \quad \tilde{F}_t'^{(n)}(\tilde{z}) = \frac{\tilde{F}_t^{(n)}(\tilde{z}) - \tilde{F}_t^{(n)}(0)}{\tilde{F}_t^{(n)}(1) - \tilde{F}_t^{(n)}(0)}.$$

Let $\tilde{\gamma} = \{0 \leq r \leq 1\}$ be the straight line from 0 to 1. If $\tilde{I}'^{(n)}(\tilde{z}) \cap \tilde{\gamma} \neq \emptyset$ for some point $\tilde{z} \in \tilde{M} \setminus \{0, 1\}$, then there exist $t_0 \in [0, 1]$ and $r_0 \in]0, 1[$ such that $\tilde{F}_{t_0}'^{(n)}(\tilde{z}) = r_0$, that is

$$(8.2) \quad \tilde{F}_{t_0}^{(n)}(\tilde{z}) - \tilde{F}_{t_0}^{(n)}(0) = r_0(\tilde{F}_{t_0}^{(n)}(1) - \tilde{F}_{t_0}^{(n)}(0)).$$

Let

$$C = \max_{i \in \{1, \dots, s\}} \sup_{t \in [0,1], \tilde{z} \in \tilde{M}} \tilde{d}(\tilde{F}_{i,t}, \tilde{z}).$$

We have

$$\left| \tilde{F}_t^{(n)}(1) - \tilde{F}_t^{(n)}(0) \right| \leq 2CN(n) + 1$$

and

$$\left| \tilde{F}_t^{(n)}(\tilde{z}) - \tilde{F}_t^{(n)}(0) \right| \geq |\tilde{z}| - 2CN(n)$$

for all $t \in [0, 1]$. Hence when $|\tilde{z}| \geq 5CN(n) + 1$, we get $\left| \tilde{F}_t'^{(n)}(\tilde{z}) \right| > 1$, i.e., $\tilde{I}'^{(n)}(\tilde{z}) \cap \tilde{\gamma} = \emptyset$.

Recall that the open disks \tilde{V} and \tilde{W} that contain ∞ in Section 4.6.1 of [39]. Here, we set $\tilde{V} = \{\tilde{z} \in \tilde{M} \mid |\tilde{z}| > 5CN(n) + 1\}$ and choose an open disk \tilde{W} containing ∞ such that $\tilde{\gamma} \cap \tilde{W} = \emptyset$, and for every $\tilde{z} \in \tilde{V}$, we have $\tilde{I}'^{(n)}(\tilde{z}) \subset \tilde{W}$. Without loss of generality, we can suppose that $z_* \notin \pi(\tilde{\gamma})$. Choose an open disk U containing z_* such that $U \cap \pi(\tilde{\gamma}) = \emptyset$. Write respectively $\tau(n, z)$ and $\Phi_n(z)$ for the first return time and the first return map of F^n throughout this proof. For every $m \geq 1$, recall that $\tau_m(n, z) = \sum_{i=0}^{m-1} \tau(n, \Phi_n^i(z))$. Let us consider the following value

$$L_m(\tilde{F}^n; 0, 1, z_*) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}'^{(n)}, z_*}^m.$$

By the same arguments with Lemma 4.6.4 and Lemma 4.6.6 in [39], we can find multi-paths $\tilde{\Gamma}_i^m(z_*)$ ($1 \leq i \leq P_m(z_*)$) from \tilde{V} to \tilde{V} such that

$$L_m(\tilde{F}^n; 0, 1, z_*) = \tilde{\gamma} \wedge \prod_{1 \leq i \leq P_m(z_*)} \tilde{\Gamma}_i^m(z_*).$$

For every $j \in \{1, \dots, s\}$ and $(\tilde{z}, \tilde{z}') \in \widetilde{M} \times \widetilde{M} \setminus \widetilde{\Delta}$, there is a unique function $\theta_j : [0, 1] \rightarrow \mathbb{R}$ such that $\theta_j(0) = 0$ and

$$e^{2i\pi\theta_j(t)} = \frac{\tilde{F}_{j,t}(\tilde{z}) - \tilde{F}_{j,t}(\tilde{z}')}{\left| \tilde{F}_{j,t}(\tilde{z}) - \tilde{F}_{j,t}(\tilde{z}') \right|}.$$

Let $\lambda_j(\tilde{z}, \tilde{z}') = \theta_j(1)$. As $\tilde{I}_j \subset \text{Diff}^1(\widetilde{M})$, there is a natural compactification of $\widetilde{M} \times \widetilde{M} \setminus \widetilde{\Delta}$ obtained by replacing the diagonal $\widetilde{\Delta}$ with the unit tangent bundle such that the map λ_j extends continuously (see, for example, [6, page 81]). Let

$$(8.3) \quad C_1 = \max_{i \in \{1, \dots, s\}} \sup_{(\tilde{z}, \tilde{z}') \in \widetilde{M} \times \widetilde{M} \setminus \widetilde{\Delta}} |\lambda_j(\tilde{z}, \tilde{z}')|.$$

Suppose that $\widetilde{M}_0 \subset \widetilde{M}$ is a closed fundamental domain with regard to the transformation group G . Denote by $\tilde{I}_i^\pm(\widetilde{M}_0)$ the set $\{\tilde{F}_{i,\pm t}(\tilde{z}) \mid (\pm t, \tilde{z}) \in [0, 1] \times \widetilde{M}_0\}$ where $\tilde{F}_{i,-t} = \tilde{F}_{i,1-t} \circ \tilde{F}_{i,1}^{-1}$. As $\tilde{F}_{i,\pm t} \circ \alpha = \alpha \circ \tilde{F}_{i,\pm t}$ for all $\alpha \in G$ and $t \in [0, 1]$, we can suppose that

$$C_2 = \max_{i \in \{1, \dots, s\}, z \in M} \sharp\{\tilde{z} \in \pi^{-1}(z) \mid \tilde{I}_i(\tilde{z}) \cap \tilde{I}_i^+(\widetilde{M}_0) \cup \tilde{I}_i^{-1}(\tilde{z}) \cap \tilde{I}_i^-(\widetilde{M}_0) \neq \emptyset\},$$

which is independent of n .

For every $0 \leq j \leq m-1$, $\tau_j(n, z_*) \leq l < \tau_{j+1}(n, z_*)$ and $1 \leq k \leq N(n)$, let $F_{i_0,1}^{\epsilon_{i_0}} = \text{Id}_{\Sigma_g}$, $\tilde{F}_{i_0,1}^{\epsilon_{i_0}} = \text{Id}_{\widetilde{M}}$,

$$z_{j,l,k} = F_{i_{k-1},1}^{\epsilon_{i_{k-1}}} (F_{i_{k-2},1}^{\epsilon_{i_{k-2}}} \circ \dots \circ F_{i_1,1}^{\epsilon_{i_1}} (F^{n(l-\tau_j(n,z_*))}(\Phi_n^j(z_*))))$$

and

$$\tilde{z}_k^0 = \tilde{F}_{i_{k-1},1}^{\epsilon_{i_{k-1}}} (\tilde{F}_{i_{k-2},1}^{\epsilon_{i_{k-2}}} \circ \dots \circ \tilde{F}_{i_1,1}^{\epsilon_{i_1}}(0)), \quad \tilde{z}_k^1 = \tilde{F}_{i_{k-1},1}^{\epsilon_{i_{k-1}}} (\tilde{F}_{i_{k-2},1}^{\epsilon_{i_{k-2}}} \circ \dots \circ \tilde{F}_{i_1,1}^{\epsilon_{i_1}}(1)).$$

When $\frac{k-1}{N(n)} \leq t \leq \frac{k}{N(n)}$, recall that

$$\tilde{F}_t^{(n)}(\tilde{z}) = \frac{\tilde{F}_{i_k, N(n)t-(k-1)}^{\epsilon_{i_k}}(\tilde{z}) - \tilde{F}_{i_k, N(n)t-(k-1)}^{\epsilon_{i_k}}(\tilde{z}_k^0)}{\tilde{F}_{i_k, N(n)t-(k-1)}^{\epsilon_{i_k}}(\tilde{z}_k^1) - \tilde{F}_{i_k, N(n)t-(k-1)}^{\epsilon_{i_k}}(\tilde{z}_k^0)}.$$

For every $\tilde{z} \in \widetilde{M}$, denote by $\tilde{J}_k(\tilde{z})$ the curve

$$\tilde{J}_k(\tilde{z}) = \left(\tilde{F}_t^{(n)}(\tilde{z}) \right)_{\frac{k-1}{N(n)} \leq t \leq \frac{k}{N(n)}}.$$

For every k , define the immersed square

$$\begin{aligned} A_k : [0, 1]^2 &\rightarrow \widetilde{M} \\ (t, r) &\mapsto \tilde{F}_{i_k, t}^{\epsilon_{i_k}}(\tilde{z}_k^0) + r(\tilde{F}_{i_k, t}^{\epsilon_{i_k}}(\tilde{z}_k^1) - \tilde{F}_{i_k, t}^{\epsilon_{i_k}}(\tilde{z}_k^0)). \end{aligned}$$

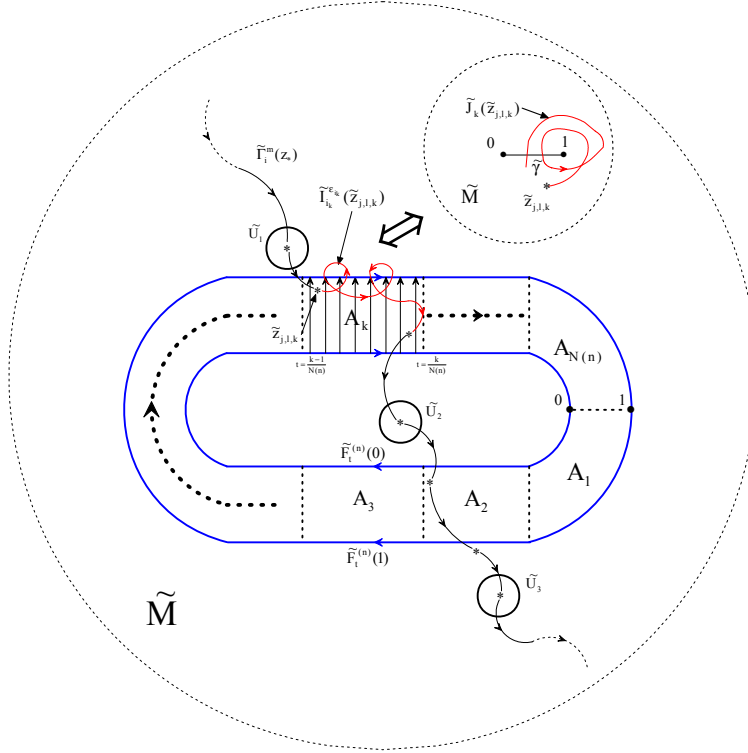


FIGURE 3. The proof of Proposition 7.1

By Equality 8.2, we know that $\tilde{J}_k(\tilde{z}) \cap \tilde{\gamma} \neq \emptyset$ implies $\tilde{I}_{i_k}^{\epsilon_{i_k}}(\tilde{z}) \cap A_k \neq \emptyset$ (see Figure 3⁴). Let

$$C_{j,l,k} = \{\tilde{z}_{j,l,k} \in \pi^{-1}(z_{j,l,k}) \mid \tilde{I}_{i_k}^{\epsilon_{i_k}}(\tilde{z}_{j,l,k}) \cap A_k \neq \emptyset\}.$$

For each k , since $|\tilde{z}_k^0 - \tilde{z}_k^1| \leq 2C(k-1) + 1$, there exists $C_3 > 0$ (depending only on \tilde{a} and \tilde{b}) such that

$$(8.4) \quad \#\{\alpha \in G \mid A_k \cap \alpha(\tilde{M}_0) \neq \emptyset\} \leq C_3 N(n).$$

Therefore,

$$(8.5) \quad \sum_{j,l,k} \#C_{j,l,k} \leq C_2 C_3 N^2(n) \tau_m(n, z_*).$$

We obtain

$$(8.6) \quad L_m(\tilde{F}^n; 0, 1, z_*) = \tilde{\gamma} \wedge \prod_{1 \leq i \leq P_m(z_*)} \tilde{\Gamma}_i^m(z_*) = \tilde{\gamma} \wedge \prod_{j,l,k} \left(\prod_{\tilde{z} \in C_{j,l,k}} \tilde{J}_k(\tilde{z}) \right).$$

We get

$$(8.7) \quad \left| L_m(\tilde{F}^n; 0, 1, z_*) \right| \leq c_0 N^2(n) \tau_m(n, z_*),$$

where $c_0 = C_1 C_2 C_3$. Therefore,

$$\left| ni(\tilde{F}; 0, 1, z_*) \right| = \left| i(\tilde{F}^n; 0, 1, z_*) \right| \leq c_0 N^2(n).$$

⁴In Figure 3, there are two universal covers that the curves in \tilde{M} (the big one) is generated by the isotopy $\tilde{I}^{(n)}$, and that the curve in \tilde{M} (the small one) is generated by \tilde{J}_k (and hence by the isotopy $\tilde{I}^{(n)}$ defined by Formula 8.1).

This implies that, for every $n \geq 1$,

$$0 < \left| i(\tilde{F}; 0, 1, z_*) \right| \leq c_0 \frac{N^2(n)}{n}.$$

That is $\|F^n\|_{\mathcal{G}} \succeq \sqrt{n}$. \square

Proof of Lemma 7.3. By the definition of $\text{Hameo}(\Sigma_g, \mu)$, we know that $\rho_{\Sigma_g, I}(\mu) \neq 0$. Assume that $F \in \mathcal{G} = \langle F_{1,1}, \dots, F_{s,1} \rangle \subset \text{Homeo}_*(\Sigma_g, \mu)$ and I_i ($1 \leq i \leq s$) are the identity isotopies corresponding to $F_{i,1}$. Recall that $\|\cdot\|_{H_1(\Sigma_g, \mathbb{R})}$ is the norm on the space $H_1(\Sigma_g, \mathbb{R})$. Write

$$\kappa = \max_{i \in \{1, \dots, s\}} \left\{ \|\rho_{\Sigma_g, I_i}(\mu)\|_{H_1(\Sigma_g, \mathbb{R})} \right\}.$$

As $\rho_{\Sigma_g, I}(\mu) \neq 0$ and $F \in \mathcal{G}$, we have $\kappa > 0$.

For every $n \in \mathbb{N}$, if I^n is homotopic to $\prod_{s=1}^{N(n)} I_{i_s}^{\epsilon_{i_s}}$, then we have

$$n \cdot \|\rho_{\Sigma_g, I}(\mu)\|_{H_1(\Sigma_g, \mathbb{R})} = \|\rho_{\Sigma_g, I^n}(\mu)\|_{H_1(\Sigma_g, \mathbb{R})} = \left\| \sum_{s=1}^{N(n)} \rho_{\Sigma_g, I_{i_s}^{\epsilon_{i_s}}}(\mu) \right\|_{H_1(\Sigma_g, \mathbb{R})} \leq \kappa \cdot N(n).$$

Hence $\|F^n\|_{\mathcal{G}} \succeq n$. On the other hand, we have $\|F^n\|_{\mathcal{G}} \preceq n$, which completes the proof. \square

Proof of Lemma 7.4. It is sufficient to prove that $\|F^n\|_{\mathcal{G}} \succeq n$. We use the notations in the proofs of Theorem 1.5 and Lemma 7.1.

If $\sharp \text{Fix}_{\text{Cont}, I}(F) = +\infty$, assume that $X \subset \text{Fix}_{\text{Cont}, I}(F)$, $I', Y = \{a, b\} \subset X, I'_Y$ are the notations defined in the proof for the case $\sharp \text{Fix}_{\text{Cont}, I}(F) = +\infty$ of Theorem 1.5. If $\sharp \text{Fix}_{\text{Cont}, I}(F) < +\infty$, for convenience, we require $a = \alpha(\lambda)$, $b = \omega(\lambda)$, and $I'_Y = I'$ where $\alpha(\lambda)$, $\omega(\lambda)$ and I' are the notions defined in the proof for the case $\sharp \text{Fix}_{\text{Cont}, I}(F) < +\infty$ of Theorem 1.5. Suppose that \tilde{I}', \tilde{I}'_Y are respectively the lifts of I' and I'_Y to \tilde{M} . Choose a lift \tilde{a} of a and a lift \tilde{b} of b . We know that $I_\mu(\tilde{F}; a, b) \neq 0$. As $F \neq \text{Id}_{\Sigma_g}$ and μ has full support, by Item **A2** in the proof of Theorem 1.5, we can choose $z_* \in \text{Rec}^+(F) \setminus X$, such that $\rho_{\Sigma_g, I}(z_*)$ and $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ exist, and $i(\tilde{F}; \tilde{a}, \tilde{b}, z_*)$ is not zero.

Suppose now that $z \in \Sigma_g \setminus X$. By the items 3 and 5 of Theorem 5.3, we know that $I(z)$ and $I'_Y(z)$ are homotopic in Σ_g . Hence, for every $n \in \mathbb{N}$, $I^{(n)}(z) = \prod_{j=1}^{N(n)} I_{i_j}^{\epsilon_{i_j}}(z)$ is homotopic to $(I'_Y)^n(z)$ in Σ_g .

Recall that $\tilde{I}^{(n)}$ is the lift of $I^{(n)}$ to \tilde{M} and \tilde{M}_0 is a closed fundamental domain with regard to G . Obviously, there exists $C' > 0$ independent of z such that

$$\sharp \{ \alpha \in G \mid \tilde{I}^{(n)}(z) \cap \alpha(\tilde{M}_0) \neq \emptyset \} \leq C' N(n).$$

Observe that $N(k) - N(1) \leq N(k+1) \leq N(k) + N(1)$ for every $1 \leq k < n$. This implies that there exists $C'_2 > 0$, independent of z , such that

$$(8.8) \quad \sharp \{ \alpha \in G \mid (\tilde{I}'_Y)^n(z) \cap \alpha(\tilde{M}_0) \neq \emptyset \} \leq C'_2 N(n).$$

Assume that $C_{j,l,k}$, $\tilde{\gamma}$, $\tilde{I}'^{(n)}$, C_1 and C_2 are the notations in the proof of Lemma 7.1.

Observing that $(\tilde{I}'_Y)^n$ and $\tilde{I}'^{(n)}$ are two isotopies from $\text{Id}_{\tilde{M}}$ to \tilde{F}^n which fix \tilde{a} and \tilde{b} , by Remark 2.4, $(\tilde{I}'_Y)^n$ is homotopic to $\tilde{I}'^{(n)}$ in $\tilde{M} \setminus \{\tilde{a}, \tilde{b}\}$.

Recall that if $z \in X \setminus \{a, b\}$, then $\gamma \wedge I'_Y(z) = 0$ (see the proof of Theorem 1.5). For every $z \in \Sigma_g \setminus X$, if there is at least one of the ends of $\tilde{I}'^{(n)}(\tilde{z})$ on $\tilde{\gamma}$, we construct a new path by extending $\tilde{I}'^{(n)}(\tilde{z})$ a little bit such that the ends of the new path are not on $\tilde{\gamma}$ and denote the new one still by $\tilde{I}'^{(n)}(\tilde{z})$. Consider the value $V(\tilde{z}) = |\tilde{\gamma} \wedge \tilde{I}'^{(n)}(\tilde{z})| + 2$. For every $z \in \Sigma_g \setminus \{a, b\}$, by Inequality 8.8, we have

$$(8.9) \quad \sharp\{\tilde{z} \in \pi^{-1}(z) \mid V(\tilde{z}) \neq 0\} \leq C'_2 N(n).$$

Write

$$\begin{aligned} C'_{j,l} &= \{\tilde{z} \in \pi^{-1}(z_{j,l,N(n)}) \mid V(\tilde{z}) > 2\}, \\ C'_{j,l,k} &= \{\tilde{z}_{j,l,k} \in \pi^{-1}(z_{j,l,k}) \mid \tilde{I}_{i_k}^{\epsilon_{i_k}}(\tilde{z}_{j,l,k}) \cap (A_k(\{r=0\} \cup \{r=1\})) \neq \emptyset\}. \end{aligned}$$

Obviously,

$$\sharp C'_{j,l,k} \leq 2C_2, \quad \sum_{j,l,k} \sharp C'_{j,l,k} \leq 2C_2 N(n) \tau_m(n, z_*).$$

Under the hypotheses of Lemma 7.4, we want to improve the value $N^2(n)$ to $N(n)$ in Inequality 8.7.

Based on the analyses above, we get

$$L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) = \tilde{\gamma} \wedge \tilde{\Gamma}_{\tilde{I}'^{(n)}, z_*}^m = \tilde{\gamma} \wedge \tilde{\Gamma}_{(\tilde{I}'_Y)^n, z_*}^m = \pi(\tilde{\gamma}) \wedge \Gamma_{(I'_Y)^n, z_*}^m.$$

To estimate the value $L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*)$, we need to consider the isotopy $\tilde{I}'^{(n)}$. If the immersed squares A_k ($k = 1, \dots, N(n)$) are uniformly bounded in n , then by the inequalities 8.4–8.7, we are done. We explain now the case where the squares A_k are not bounded in n , that is, $\max_{1 \leq k \leq N(n)} |\tilde{z}_k^0 - \tilde{z}_k^1| \rightarrow +\infty$ as $n \rightarrow +\infty$, is also true. Inequality 8.9 shows that, for every fixed j and l , it is sufficient to consider at most $C'_2 N(n)$ elements of $C'_{j,l}$. We can write $\tilde{I}'^{(n)}(\tilde{z})$ as the concatenation of $N(n)$ sub-paths $\tilde{J}_k(\tilde{z})$ ($k = 1, \dots, N(n)$). Obviously, for every $\tilde{z} \in C'_{j,l}$, we have

$$(8.10) \quad \left| L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) \right| \leq \left| \tilde{\gamma} \wedge \prod_{j,l,k} \left(\prod_{\tilde{z} \in C'_{j,l,k}} \tilde{J}_k(\tilde{z}) \right) \right| + \left| \tilde{\gamma} \wedge \prod_{j,l} \left(\prod_{\tilde{z} \in C'_{j,l}} \tilde{I}'^{(n)}(\tilde{z}) \right) \right|.$$

We know that the value of the first part of the right hand side of Inequality 8.10 is less than $2C_1 C_2 N(n) \tau_m(n, z_*)$. Hence, to explore the relation of the bound of $\left| L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) \right|$ and the power of $N(n)$, we can suppose that the path $\tilde{J}_k(\tilde{z})$ never meets $A_k(\{r=0\} \cup \{r=1\})$ for every k and $\tilde{z} \in C'_{j,l}$. As the isotopies \tilde{I}_i ($1 \leq i \leq s$) commutes with the covering transformations, it is easy to prove that there is a positive number C'_1 such that

$$\sum_{z \in C'_{j,l}} V(\tilde{z}) \leq C'_1 C'_2 N(n).$$

Hence,

$$\begin{aligned} \left| \tilde{\gamma} \wedge \left(\prod_{j,l} \prod_{\tilde{z} \in C'_{j,l}} \tilde{I}'^{(n)}(\tilde{z}) \right) \right| &= \left| \tilde{\gamma} \wedge \left(\prod_{j,l} \prod_{\tilde{z} \in C'_{j,l}} \prod_{k=1}^{N(n)} \tilde{J}_k(\tilde{z}) \right) \right| \\ &\leq C'_1 C'_2 N(n) \tau_m(n, z_*). \end{aligned}$$

Therefore, there exists $C''_1 > \max\{C_1, C'_1\}$, where C_1 defined in Formula 8.3, such that

$$\left| L_m(\tilde{F}^n; \tilde{a}, \tilde{b}, z_*) \right| \leq C''_1 (2C_2 + C'_2) N(n) \tau_m(n, z_*).$$

It implies that

$$0 < \left| i(\tilde{F}; \tilde{a}, \tilde{b}, z_*) \right| \leq c'_0 \frac{N(n)}{n} \quad \text{for every } n \geq 1,$$

where $c'_0 = C''_1(2C_2 + C'_2)$. We deduce that $\|F^n\|_{\mathcal{G}} \geq n$. Therefore, $\|F^n\|_{\mathcal{G}} \sim n$, which completes the proof. \square

8.4. Counter-examples. We construct two examples for $\text{Supp}(\mu) \neq M$.

Example 8.3. Consider the following smooth identity isotopy on \mathbb{R}^2 : $\tilde{I} = (\tilde{F}_t)_{t \in [0,1]} : (x, y) \mapsto (x + \frac{t}{10} \cos(2\pi y), y + \frac{t}{10} \sin(2\pi y))$. It induces an identity smooth isotopy $I = (F_t)_{t \in [0,1]}$ on \mathbb{T}^2 . Let μ have a constant density on $\{(x, y) \in \mathbb{T}^2 \mid y = 0 \text{ or } y = \frac{1}{2}\}$ and vanish on elsewhere. Obviously, $\rho_{\mathbb{T}^2, I}(\mu) = 0$ but $\text{Fix}_{\text{Cont}, I}(F_1) = \emptyset$.

In the case where $g = 1$ and $\text{Supp}(\mu) \neq M$, Example 8.3 tells us that there is even no sense to talk about the action function in some special cases. The following example, belonging to Le Calvez, implies that Theorem 1.5 is no longer true in the case where $g > 1$ and $\text{Supp}(\mu) \neq M$.

Example 8.4 ([24], page 73). Let M be a closed orientated surface with $g = 2$. Le Calvez constructed a smooth identity isotopy $I = (F_t)_{t \in [0,1]}$, two points a and b (z_3 and z_4 respectively in his example) which are the only two contractible fixed points of F_1 , and a point c ($z'_{1,3}$ in his example) which is a periodic point of F_1 with period 20 and an arc $\prod_{0 \leq i \leq 19} I(F_1^i(c))$ which is homologic to zero in $M \setminus \{a, b\}$.

We now define the measure $\mu = \frac{1}{20} \sum_{i=0}^{19} \delta_{F_1^i(c)}$, where δ_z is the Dirac measure. The fact that the arc $\prod_{0 \leq i \leq 19} I(F_1^i(c))$ is homologic to zero in $M \setminus \{a, b\}$ means that $\rho_{M, I}(\mu) = 0$ and that $I_\mu(\tilde{F}; a, b) = 0$, i.e., the action function is constant.

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MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTRASSE 22, D-04103 LEIPZIG, GERMANY

CHERN INSTITUTE OF MATHEMATICS AND KEY LAB OF PURE MATHEMATICS AND COMBINATORICS OF MINISTRY OF EDUCATION, NANKAI UNIVERSITY, TIANJIN 300071, P.R.CHINA

E-mail address: jianwang@mis.mpg.de; wangjian@nankai.edu.cn